127. On Schwarz's Lemma for $\Delta u+c(x)u=0$

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1. **Introduction.** The famous Schwarz's lemma in the complex function theory states that if $f(z)$ is holomorphic in $|z|<1$, and if $f(0)=0$ (or $|f(z)|\leq \text{const.}\,|z|$), then the estimate: $|f(z)|\leq|z|\max_{|\zeta|=1}|f(\zeta)|$ holds for $|z|<1$. Many theorems in the complex function theory have been generalized with great success to the case of harmonic functions, or more generally, solutions of the second order elliptic differential equations with variable coefficients. The Schwarz's lemma, however, does not seem to have been generalized previously even to the case of harmonic functions. The main purpose of the present paper is to generalize this lemma to solutions of the equations of the form: $\Delta u+c(x)u=0$. As corollaries of the generalized Schwarz's lemma, we can obtain the generalizations of the Hadamard three-circles theorem [which states; if $f(z)$ is holomorphic in $|z|<R$, then $\log\max_{\theta}|f(re^{i\theta})|$ is a convex function of $\log r(0<r<R)$], and the Liouville's theorem [which states; if an entire function $f(z)$ satisfies $0(|z|^k)$ as $|z|\to\infty$ (k; non-negative integer), then $f(z)$ is a polynomial of at most degree k]. The extension of the results below to the case of the general second order elliptic equations with variable coefficients will be published elsewhere.

2. **Notations.** R^n denotes the n-dimensional real Euclidean space, and C^n the n-dimensional complex Euclidean space. We denote the inner product and the norm in C^n (R^n) by $\langle\cdot,\cdot\rangle$ and $|\cdot|$. We set $S_R=\{x\in R^n;\,0<|x|<R\}$. Let $L^2(\Sigma)$ be the L^2-space of C^m-valued functions defined on the unit surface $\Sigma=\{x\in R^n;\,|x|=1\}$. Then $L^2(\Sigma)$ is the Hilbert space with the usual inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$: $(u,v)=\int_\Sigma\langle u(\xi),v(\xi)\rangle d\sigma_\xi$ and $\|u\|=(u,u)^{1/2}$ where $d\sigma_\xi$ denotes the surface area element on the unit surface Σ. $H^2(\Sigma)$ denotes the set of all functions in $L^2(\Sigma)$ whose distribution derivatives up to order 2 belong to $L^2(\Sigma)$. Now putting $u(r,\xi)=u(x)$ ($r=|x|$; $\zeta=x/|x|$), we shall then regard a function $u(x)$ defined in the sphere $|x|<R$ as an $L^2(\Sigma)$-valued function of r, and simply write $u(r)$ for $u(r,\xi)$ (or $u(x)$). Finally, for multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $x=(x_1,\cdots,x_n)\in R^n$, we set $x^\alpha=x_1^{\alpha_1}\cdots\alpha_n^{\alpha_n}$ and $D^\alpha=\partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$.