124. *Hypoelliptic Differential Operators with Double Characteristics*

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In this note, we shall consider the hypoellipticity of the following operator in $\boldsymbol{R}^2$:

$$P(x,t,D_x,\partial_t)=(\partial_t+taD_x)(\partial_t+tbD_x)+cD_x + A(x,t)tD_x+B(x,t),$$

where $\partial_t=\partial/\partial t$, $D_x=-i\partial/\partial x$ and $a,b,c\in \boldsymbol{C}$ and $A(x,t)$, $B(x,t)\in C^\infty(\boldsymbol{R}^2)$. (Cf. Grušin [1], [2], Sjöstrand [3], Treves [4].) A linear (pseudo-) differential operator $Q(x,D_x)$ in $\boldsymbol{R}^n$ is called hypoelliptic in an open subset $\Omega\subset \boldsymbol{R}^n$ if

$$\text{sing supp}\, u=\text{sing supp}\, Qu, \qquad u\in \mathcal{E}'(\Omega).$$

If $A\equiv 0$ and $B\equiv 0$, then we have

Theorem 0 (cf. [1], Theorem 1.2). *Assume that* $\mathrm{Re}\, a\cdot \mathrm{Re}\, b<0$. *Then*

$$P_1(x,t,D_x,\partial_t)=(\partial_t+taD_x)(\partial_t+tbD_x)+cD_x$$

is hypoelliptic in $\boldsymbol{R}^2$ *if and only if*

$$\frac{c}{b-a}\notin \boldsymbol{Z}.$$

Thus, in this note, we assume that

$$\text{(A)} \qquad \mathrm{Re}\, a<0, \mathrm{Re}\, b>0, \frac{c}{b-a}\in \boldsymbol{Z}^+\cup\{0\}.$$

We shall give the *sufficient* conditions on A, B for P to be hypoelliptic in a neighbourhood of $(x,t)=(0,0)$ (see Corollary 1 and Corollary 2 below). The case that $\mathrm{Re}\, a>0$, $\mathrm{Re}\, b<0$, $c/(b-a)\in \boldsymbol{Z}^+\cup\{0\}$ can be proved in exactly the same way. Now we state the main result:

Theorem 1 (cf. [3], Proposition 5.4). *Under the assumption* (A), *there exist properly supported operators*

$$\mathcal{P}=\begin{pmatrix} P, & R^- \\ R^+, & 0 \end{pmatrix}: \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus \\ \mathcal{D}'(\boldsymbol{R})\end{matrix} \rightarrow \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus \\ \mathcal{D}'(\boldsymbol{R})\end{matrix}$$

$$\mathcal{G}=\begin{pmatrix} G, & G^+ \\ G^-, & G^{-+} \end{pmatrix}: \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus \\ \mathcal{D}'(\boldsymbol{R})\end{matrix} \rightarrow \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus \\ \mathcal{D}'(\boldsymbol{R})\end{matrix}$$

with the following properties:

(i) $\mathcal{G}\cdot\mathcal{P}-I$ *and* $\mathcal{P}\cdot\mathcal{G}-I$ *have* C^∞ *kernels.*

(ii) *For all* $s\in \boldsymbol{R}$

$$G: H_s^{\mathrm{loc}}(\boldsymbol{R}^2)\rightarrow H_{s+1}^{\mathrm{loc}}(\boldsymbol{R}^2),$$