121. *Kähler Metrics on Elliptic Surfaces*

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The purpose of this note is to outline a proof of the following

Theorem. *An elliptic surface admits a Kähler metric if and only if its first Betti number is even.*

Professor Kodaira raised a problem: *Does every compact analytic surface with an even first Betti number admit a Kähler metric?*

Our theorem solves this problem in the affirmative except the case in which the surface is a $K3$ surface.

1. Some cohomology groups on elliptic surfaces. Let $\Phi: B\to\Delta$ be an elliptic surface with a section $o: \Delta\to B$. We employ the notation of Kodaira [2]. Thus J, G and $\mathfrak{f}$ denote, respectively, the functional invariant of B, the homological invariant of B and the normal bundle of $o(\Delta)$ in B.

The following proposition is due to Shioda [5].

Proposition 1. *There exist canonical homomorphisms*

$$\alpha: H^1(\Delta, G)\to j^*(H^2(B, \boldsymbol{Z}))\subset H^2(B, \mathcal{O}),$$
$$\beta: H^1(\Delta, \mathcal{O}(\mathfrak{f}))\to H^2(B, \mathcal{O}),$$

such that

(i) $\operatorname{Im}\alpha$ *is a commensurable subgroup of* $j^*(H^2(B, \boldsymbol{Z}))$,

(ii) β *is an isomorphism,*

(iii) *the diagram*

$$\begin{array}{ccc} H^1(\Delta, G) & \xrightarrow{i^*} & H^1(\Delta, \mathcal{O}(\mathfrak{f})) \\ \alpha\downarrow & & \downarrow\beta \\ j^*(H^2(B, \boldsymbol{Z})) & \hookrightarrow & H^2(B, \mathcal{O}) \end{array}$$

is commutative, where i^* and j^* denote the natural homorphisms induced by the canonical injections $i: G\to\mathcal{O}(\mathfrak{f})$ and $j: \boldsymbol{Z}\to\mathcal{O}$, respectively.

Proof. We have canonical isomorphisms

$$G\xrightarrow{\approx}R^1\Phi_*(\boldsymbol{Z}),$$
$$\mathcal{O}(\mathfrak{f})\xrightarrow{\approx}R^1\Phi_*(\mathcal{O}_B),$$

and, moreover, i is compatible with j^* through the isomorphisms. We shall identify G, $\mathcal{O}(\mathfrak{f})$ and i, respectively, with $R^1\Phi_*(\boldsymbol{Z})$, $R^1\Phi_*(\mathcal{O}_B)$ and j^*. Let us consider the Leray spectral sequences:

$$'E_2^{pq}=H^p(\Delta, R^q\Phi_*(\boldsymbol{Z}))\Rightarrow H^{p+q}(B, \boldsymbol{Z}),$$
$$''E_2^{pq}=H^p(\Delta, R^q\Phi_*(\mathcal{O}_B))\Rightarrow H^{p+q}(B, \mathcal{O}_B).$$