167. *Normal Expectations and Crossed Products of von Neumann Algebras*

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In this paper, we shall show that a von Neumann algebra $\mathcal{M}$ is isomorphic to the crossed product $G\otimes\mathcal{A}$ of a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ by a group G of automorphisms of $\mathcal{A}$ implemented by a unitary group in $\mathcal{M}$ under certain conditions. This result is a generalization of two theorems of Golodets [3].

1. Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let G be a discrete group of (*-) automorphisms of $\mathcal{A}$.

On the Hilbert space $\mathfrak{H}\otimes l^2(G)$, the tensor product of $\mathfrak{H}$ and $l^2(G)$, define a representation I of $\mathcal{A}$ by

$$I(A)\Big(\sum_{g\in G}\xi_g\otimes\varepsilon_g\Big)=\sum_{g\in G}g^{-1}(A)\xi_g\otimes\varepsilon_g,$$

for each A in $\mathcal{A}$ and ξ_g in $\mathfrak{H}$, where ε_g is an orthonormal basis in $l^2(G)$ such that

$$\varepsilon_g(h)=\begin{cases}1 & (g=h)\\ 0 & (g\neq h),\end{cases}\qquad g,h\in G.$$

Letting G act as a permutation group in $\mathfrak{H}\otimes l^2(G)$, we obtain a unitary representation V_g of G such that

$$V_g\Big(\sum_{h\in G}\xi_h\otimes\varepsilon_h\Big)=\sum_{h\in G}\xi_{g^{-1}h}\otimes\varepsilon_h,\qquad g\in G,\ \xi_h\in\mathfrak{H}.$$

One can then verify that I is a faithful normal representation with the covariance formula

$$V_gI(A)V_g^*=I(g(A)),\qquad g\in G,\ A\in\mathcal{A}.$$

Then the von Neumann algebra acting on $\mathfrak{H}\otimes l^2(G)$ generated by $I(\mathcal{A})$ and V_G is called the *crossed product* $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G.

Theorem 1. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$ and G a discrete group of automorphisms of $\mathcal{A}$. Assume that $(\mathcal{M},\mathcal{A},G)$ satisfies the following three conditions;*

(1) *there is a unitary representation U_g of G into $\mathcal{M}$ with $g(A)=U_gAU_g^*$ for g in G and A in $\mathcal{A}$,*

(2) *$\mathcal{M}$ admits a cyclic vector ξ with $(U_gA\xi,\xi)=0$ for $g(\neq 1)$ in G and A in $\mathcal{A}$,*

and

(3) *$\mathcal{M}$ is generated by $\mathcal{A}$ and U_G.*