161. On a Theorem of Collingwood

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We denote by D the unit disk $|z|<1$ and by Γ its boundary $|z|=1$. The *cluster set* $C_D(f, e^{i\theta})$ of a complex-valued function $f(z)$ on D at $e^{i\theta}\in\Gamma$ is the set of values α in the extended plane such that $\lim_{n\to\infty} f(z_n)=\alpha$ for a sequence $\{z_n\}$ in D convergent to $e^{i\theta}$. Let $H=\{H(\theta)\}_{e^{i\theta}\in\Gamma}$ be a family of subsets $H(\theta)$ of Γ such that

(1) $$e^{i\theta}\in H'(\theta)$$

for each $e^{i\theta}\in\Gamma$, where $H'(\theta)$ is the derived set of $H(\theta)$. We define the *boundary cluster set* $C_{\Gamma H}(f, e^{i\theta})$ relative to the family H at $e^{i\theta}\in\Gamma$ by

(2) $$C_{\Gamma H}(f, e^{i\theta})=\bigcap_{\eta>0} M_\eta(\theta)$$

where we set

(3) $$M_\eta(\theta)=\overline{\bigcup_{e^{i\theta'}\in H(\theta),0<|\theta'-\theta|<\eta} C_D(f, e^{i\theta'})}.$$

Here and hereafter the bar indicates the closure. We stress that no regularity of $f(z)$ such as holomorphy or continuity is postulated. The purpose of this note is to prove the following

Theorem. *The boundary cluster set $C_{\Gamma H}(f, e^{i\theta})$ coincides with the ordinary cluser set $C_D(f, e^{i\theta})$ for every $e^{i\theta}$ in Γ except possibly for a set of category I in the Baire sense.*

This is originally obtained by Collingwood [1], [2] for a family $H=\{H(\theta)\}_{e^{i\theta}\in\Gamma}$ such that

(4) $$H(\theta)\supset\{e^{i\theta'}; \theta'\in(\theta-\eta,\theta)\} \qquad [\{e^{i\theta'}\in(\theta,\theta+\eta)\}, \text{resp.}]$$

for an $\eta>0$ for every $e^{i\theta}\in\Gamma$. Clearly (4) implies (1), and our theorem generalizes that of Collingwood.

Proof of Theorem. Suppose that $\mathcal{M}(\theta)=\{e^{i\theta}; C_{\Gamma H}(f, e^{i\theta}) \neq C_D(f, e^{i\theta})\}$ is of category II. Let $\{\varepsilon_n\}$ be a monotonically decreasing sequence of positive numbers such that $\lim_{n\to\infty}\varepsilon_n=0$. We denote by $C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_n}$ the closed set of points whose spherical distances from $C_{\Gamma H}(f, e^{i\theta})$ are at most equal to ε_n. Let $E_n=\{e^{i\theta}; C_D(f, e^{i\theta}) - C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_n}\neq\phi\}$. Obviously $\mathcal{M}(\theta)=\cup E_n$. Hence there exists at least one N such that E_N is of category II. Let $\{\Delta_1, \Delta_2, \cdots, \Delta_m\}$ be a finite triangulation of Riemann-sphere such that each Δ_μ has a spherical diameter less than $\varepsilon_N/4$. For each μ, let $E_{N,\mu}=\{e^{i\theta}\in E_N; \{C_D(f, e^{i\theta}) - C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_N}\}\cap\bar{\Delta}_\mu\neq\phi\}$. Then $E_N=\bigcup_{1\leq\mu\leq m}E_{N,\mu}$ and therefore there exists at least one M $(1\leq M\leq m)$ such that $E_{N,M}$ is of category II.