192. *No Free Inverse Semigroup is Orderable*[*)]

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A semigroup S is said to be orderable if S admits a simple order to make S a simply ordered semigroup. It is known that every free group is orderable ([1] or [4]). On the other hand, it is shown in [3] that a free commutative idempotent semigroup with r free generators is orderable if and only if $r \leqq 2$. In this paper we consider the orderability of free inverse semigroups. Since it is easily shown that a free inverse semigroup with r free generators contains a free commutative idempotent subsemigroup with r free generators, it follows from the above result that a free inverse semigroup with a set of free generators consisting of more than three elements is not orderable. But we have the following stronger result.

Theorem. *No free inverse semigroup is orderable.*

Proof. By way of contradiction we assume that S is a free inverse semigroup which is at the same time an ordered semigroup. We denote by $\preceq$ the natural partial order on the inverse semigroup S. Let F be a set of free generators of S and let $x \in F$. Then, by [5] Corollary 2.5, the inverse subsemigroup $\langle x \rangle$ generated by x is a free inverse semigroup with a free generator x. Also in S the set E of idempotents of S forms a semilattice with respect to the partial order $\preceq$ and, by [6] Theorem 3, the semilattice E is a tree semilattice. Now we have

$$x^2x^{-2} \preceq xx^{-1}, \qquad xx^{-2}x \preceq xx^{-1}$$

and so x^2x^{-2} and $xx^{-2}x$ are comparable. If $x^2x^{-2} \preceq xx^{-2}x$, then $x^2x^{-2} \preceq xx^{-2}x \preceq x^{-1}x$, which contradicts [5] Corollary 2.4. Next we suppose $xx^{-2}x \preceq x^2x^{-2}$. Then $xx^{-2}x = (x^2x^{-2})(xx^{-2}x) = x^2x^{-3}x$ and so $x^2 = x(xx^{-2}x)x = x^3x^{-3}x^2$. Hence

$$x^{-2}x^2 = x^{-2}x^3x^{-3}x^2 = (x^{-2}x^2)(xx^{-1})(x^{-2}x^2) \preceq xx^{-1},$$

which contradicts again [5] Corollary 2.4.

References

[1] G. Birkhoff: Review of "Everett, C. J., and Ulam, S., Ordered groups." Math. Rev., **7**, 4 (1946).
[2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Vol. I. Amer. Math. Soc. (1961); Vol. II. Amer. Math. Soc. (1967).

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*) Dedicated to Professor Kiiti Morita on his 60th birthday.