

191. A Note on the Normal Subgroups of 4-fold Transitive Permutation Groups of Degree $5m+4$

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(Comm. by Kenjiro SHODA, M. J. A., Dec. 12, 1974)

1. Introduction. The following is the well-known theorem of Jordan ([8], p. 65).

Theorem (Jordan). *Let G be a k -transitive permutation group on a set Ω , $k \geq 2$, and G be not the symmetric group on Ω . If $H \neq 1$ is a normal subgroup of G , then H is $(k-1)$ -transitive on Ω with the exception when $|\Omega|$ is a power of 2 and $k=3$, and in this exceptional case H may be an elementary abelian 2-group transitive and regular on Ω .*

This theorem is refined by A. Wagner [7], N. Ito [4], M. Aschbacher [1], J. Saxl [6], and E. Bannai [2]. In this note we shall prove the following:

Theorem.¹⁾ *Let G be a 4-fold transitive permutation group on a set Ω , where $|\Omega|=5m+4$ (m : integer) and greater than 5. If $H \neq 1$ is a normal subgroup of G , then H is also 4-fold transitive on Ω .*

2. Proof of the theorem. In order to prove the theorem we need the following:

Lemma. *Let G be a t -fold transitive permutation group on a set Ω for $t \geq 4$ and let $H \neq 1$ be a normal subgroup of G . Then for all $\Delta \subseteq \Omega$ with $|\Delta|=t$, $H_{\Delta}^{\Delta} = S_t$.*

Proof. We omit the proof of the lemma. (See [3].)

Now we start with the proof of the theorem. Suppose $|\Omega|=5m+4$ be the minimal degree >5 such that there is a counterexample to the theorem. Let G be a 4-fold transitive group on Ω of degree $5m+4$ containing a non-trivial normal subgroup H which is 3-fold transitive but not 4-fold transitive on Ω . Let $\alpha, \beta, \gamma \in \Omega$ and let $\Gamma_1, \dots, \Gamma_k$ be the $H_{\alpha\beta\gamma}$ -orbits on $\Omega - \{\alpha, \beta, \gamma\}$. Then by assumption $k \geq 2$, $|\Gamma_1| = \dots = |\Gamma_k|$ and $|\Gamma_1|$ is not divisible by 5. At first we shall show that there exists a non-trivial 5-element in H fixing at least 4 points of Ω . By the lemma above and a result of Wagner ([7], Lemma 4), $|\Gamma_1|$ must be even. Let $\delta \in \Gamma_1$ and let T be a Sylow 2-subgroup of $H_{\alpha\beta\gamma\delta}$. Then $T \neq 1$ by Theorem 2 of J. King [5]. Now if $T^g \subseteq G_{\alpha\beta\gamma\delta}$ for some $g \in G$ then $T^g \subseteq G_{\alpha\beta\gamma\delta} \cap H = H_{\alpha\beta\gamma\delta}$ and hence there is an element h of $H_{\alpha\beta\gamma\delta}$ such that $T^g = T^h$. Thus by a well-known lemma of Witt $N_G(T)^{f^t x T}$ is 4-fold transitive.

1) I was informed that the same result was obtained also by E. Bannai (University of Tokyo) independently.