189. On Spaces which Admit Closure-Preserving Covers by Compact Sets

By Yûkiti KATUTA

(Comm. by Kenjiro SHODA, M. J. A., Dec. 12, 1974)

Let X be a T_1-space which admits a closure-preserving closed cover $\mathcal{F}$ by compact subsets. In [9], H. Tamano raised the question of whether or not such a space X must be paracompact. H. B. Potoczny gave in [6] a negative answer to this question, and proved in [7] that such a space X is paracompact whenever it is collectionwise normal.

In the same paper [7], he has stated that if each member of $\mathcal{F}$ is a finite subset then X is θ-refinable, and if, in addition, there is a positive integer n such that each member of $\mathcal{F}$ has no more than n points then X is metacompact. Moreover, he has conjectured that X must be always metacompact or θ-refinable without such severe restrictions.

In this paper, we shall give a solution to this problem.

Theorem 1. *If a T_1-space X has a closure-preserving closed cover $\mathcal{F}$ by compact subsets, then X is metacompact.*

It is known that a metacompact, collectionwise normal space is paracompact ([3] or [5]); consequently the above result of Potoczny follows immediately from our Theorem 1.

We need some lemmas to prove Theorem 1. A space X is said to be *almost expandable* [8], if for every locally finite collection $\{F_\alpha | \alpha \in A\}$ of subsets of X there exists a point-finite collection $\{G_\alpha | \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha \in A$; every metacompact space is almost expandable ([8]). A cover $\mathcal{U}$ of a space is said to be *directed*, if for two members U and V of $\mathcal{U}$ there is a member W of $\mathcal{U}$ such that $U \cup V \subset W$.

The following lemma is an immediate consequence of [2, Theorem 2.2] (announced in [1]).

Lemma 1. *If every directed open cover of a space X has a cushioned refinement, then X is almost expandable.*

Lemma 2. *If a space X has a closure-preserving closed cover $\mathcal{F}$ by compact subsets, then X is almost expandable.*

Proof. Let $\mathcal{U}$ be a directed open cover of X. Since each member of $\mathcal{F}$ is compact, we can easily prove that $\mathcal{F}$ refines $\mathcal{U}$. Furthermore $\mathcal{F}$ is a closure-preserving closed cover, so that $\mathcal{F}$ is a cushioned refinement of $\mathcal{U}$ ([4]). Hence, by Lemma 1, X is almost expandable.