# 184. *A Remark on Picard Principle*

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A nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leq 1$ will be referred to as a *density* on the punctured unit disk $\Omega: 0<|z|<1$ with a singularity at $z=0$, removable or genuine. The *elliptic dimension* of a density $P$ on $\Omega$ at $z=0$, $\dim P$ in notation, is the dimension of the half module of nonnegative solutions $u$ of the equation $\Delta u=Pu$ on $\Omega$ with vanishing boundary values on $|z|=1$. After Bouligand we say that the *Picard principle* is valid for a density $P$ at $z=0$ if $\dim P=1$. That the Picard principle is valid for the density $P(z)\equiv 0$, i.e. for the harmonic case, is the well known classical result. Less trivial examples are $P(z)=|z|^{-\lambda}$ ($\lambda\in(-\infty,2]$) (cf. [2]) and densities $P(z)$ with the property

$$\int_{\Omega-E} P(z)\log\frac{1}{|z|}dxdy<\infty \qquad (z=x+iy)$$

where $E=E_P$ is a closed subset of $\Omega$ thin at $z=0$ (cf. [3]). These examples suggest that singularities of densities $P(z)$ at $z=0$ for which the Picard principle is valid are 'not so wild'. In view of this one might be tempted to say that if the Picard principle is valid for two densities $P_j$ ($j=1,2$), then it is also valid for the density $P_1+P_2$. The purpose of this note is to stress the complexity of the Picard principle by showing that the above intuition is wrong. Namely we shall prove the following

**Theorem.** *There exists a pair of densities $P_j$ ($j=1,2$) on $\Omega$ such that the Picard principle is valid for $P_j$ ($j=1,2$) at $z=0$ but invalid for the density $P_1+P_2$ at $z=0$.*

Actually densities $P_j$ ($j=1,2$) we are going to construct as stated in the above theorem are *rotation free* in the sense that $P_j(z)=P_j(|z|)$ on $\Omega$, and satisfy $\dim P_j=1$ ($j=1,2$) and $\dim(P_1+P_2)=\mathfrak{c}$ (the cardinal number of continuum). This also shows the invalidity of subadditivity of elliptic dimensions, i.e. the following inequality does not hold in general:

$$\dim(P_1+P_2)\leq\dim P_1+\dim P_2.$$

**1.** To construct the required $P_j$ we need to consider auxiliary functions $s(t;\lambda,\mu)$ and $c(t;\lambda,\mu)$ which are modifications of trigonometric functions. Let $\lambda\in[1,2)$ and $\mu\in\boldsymbol{R}$ (the real number field). Consider mutually disjoint closed intervals