179. On the Existence Proof of Haar Measure

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H. Cartan [1] established the existence of the Haar measure on arbitrary locally compact topological groups without the axiom of choice. In that case, the method used by him was to construct the Haar measure as a limit based on Cauchy's convergence criterion for filter. The purpose of this paper is to construct, following the idea of Cartan, the Harr measure by the method of ranked spaces. In this case, the Haar measure is given as a limit based on Cauchy's convergence criterion for "sequence".

Throughout this paper, terminologies and notations concerning ranked spaces are the same as in [2].

We begin with the definition of the ranked space $(\mathcal{K}, \mathcal{V})$ needed for our proof.

1. Definition of the ranked space $(\mathcal{K}, \mathcal{V})$. Let $\mathcal{K}$ be the vector space consisting of all continuous real-valued functions $f \geq 0$ on G with compact supports, and let $\boldsymbol{K}$ be the family of all compact subsets of G. For each $K \in \boldsymbol{K}$, we denote by $\mathcal{K}(K)$ the vector subspace of $\mathcal{K}$ formed of all functions whose supports are contained in K, and denote by $(\mathcal{K}(K), d)$ the metric space $\mathcal{K}(K)$ provided with the distance function $d(f, g) = \|f - g\|$, where $\| \ \|$ is the uniform norm. We denote the ranked union space of the ranked metric spaces $(\mathcal{K}(K), \mathcal{V}(d))$ $(K \in \boldsymbol{K})$ by $(\mathcal{K}, \mathcal{V})$.

We first know that the following holds for $(\mathcal{K}, \mathcal{V})$ by [2], Theorem 1 and Proposition 2, since $(\mathcal{K}(K), \mathcal{V}(d))$ $(K \in \boldsymbol{K})$ satisfies the condition $(\dagger_1)$ in [2].

Proposition 1. *We have* $\{r\text{-}\lim f_n\} \ni f$ *in* $(\mathcal{K}, \mathcal{V})$ *if and only if there is a* $K \in \boldsymbol{K}$ *such that* $\operatorname{supp} f \subset K$ *and* $\operatorname{supp} f_n \subset K$ $(n=0, 1, 2, \cdots)$, *and we have* $\lim d(f_n, f) = 0$.

For $s \in G$ and $f \in \mathcal{K}$, we denote the left translation of f by $s: f(s^{-1}x)$ by sf, and the support of f by $\operatorname{supp} f$. Furthermore we denote the family of all functions f^* expressed as $f^* = \sum_{i=1}^{i_0} c_i s_i g$, where $g \in \mathcal{K}$ and $c_i \geq 0$, $s_i \in G$ $(i=1, 2, \cdots, i_0)$, by $\mathcal{K}_0$, and denote g by $\chi(f^*)$ and $\sum_{i=1}^{i_0} c_i$ by $c(f^*)$ for such f^*.

2. Results already known (cf. [1]). We fix a function $f_0 \in \mathcal{K}$ with $f_0 \neq 0$, and for $f, \varphi \in \mathcal{K}$ with $\varphi \neq 0$, we put

$$(f : \varphi) = \inf \{c(f^*) ; f \leq f^* (\in \mathcal{K}_0), \chi(f^*) = \varphi\},$$
$$I_\varphi(f) = (f : \varphi)/(f_0 : \varphi).$$