8. *Universal Sentences Preserved under Certain Extensions*

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The purpose of this paper is to characterize the universal sentences preserved under the formation of zero-element extensions. Here a zero-element extension is defined as follows.

Let $\mathfrak{A}=\langle A, \{f_\xi \mid \xi<\rho\}, \{r_\eta \mid \eta<\sigma\}\rangle$ and $\mathfrak{A}^*=\langle A^*, \{f_\xi^* \mid \xi<\rho\}, \{r_\eta^* \mid \eta<\sigma\}\rangle$ be structures of the same similarity type, where A, A^* are domains, f_ξ, f_ξ^* are $n(\xi)$-ary operations, and r_η, r_η^* are $m(\eta)$-ary relations. $\mathfrak{A}^*$ is called a *zero-element extension* of $\mathfrak{A}$ if the following three conditions hold:

(1) A^* consists of all elements in A and an element not contained in A which is denoted by o, i.e. $A^*=A\cup\{o\}$ and $o\notin A$;

(2) For any $\xi<\rho$ and any $a_1, \cdots, a_{n(\xi)}\in A^*$,

$$f_\xi^*(a_1, \cdots, a_{n(\xi)})=\begin{cases} a & \text{if } f_\xi(a_1, \cdots, a_{n(\xi)})=a, \\ o & \text{if at least one of } a_1, \cdots, a_{n(\xi)} \text{ is } o; \end{cases}$$

(3) For any $\eta<\sigma$ and any $a_1, \cdots, a_{m(\eta)}\in A^*$, $r_\eta^*(a_1, \cdots, a_{m(\eta)})$ if and only if either $r_\eta(a_1, \cdots, a_{m(\eta)})$ or $a_1=\cdots=a_{m(\eta)}=o$.

Each of the well-known preservation theorems asserts that a sentence is preserved under a given algebraic construction (or constructions) if and only if it is equivalent to a sentence having certain formal properties which depend chiefly on occurrences of logical symbols. However, the formal properties of sentences, which appear in our discussion, depend largely on occurrences of individual variables as well as occurrences of logical symbols.

Let L be a first-order language with or without equality. A structure $\mathfrak{A}$ of the similarity type corresponding to L is simply called a structure for L. The domain of $\mathfrak{A}$ is denoted by $D[\mathfrak{A}]$. Let Φ be any formula of L which contains at most some of the distinct variables $x_1, \cdots, x_n$ as free variables, and let $a_1, \cdots, a_n$ be elements in $D[\mathfrak{A}]$. Then we write $\mathfrak{A}\models\Phi[a_1/x_1, \cdots, a_n/x_n]$, if $a_1, \cdots, a_n$ satisfy Φ in $\mathfrak{A}$ when the free variables $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively. If $\mathfrak{A}\models\Phi[a_1/x_1, \cdots, a_n/x_n]$ holds for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, we say that Φ is valid in $\mathfrak{A}$, and we write $\mathfrak{A}\models\Phi$. If $\mathfrak{A}\models\Phi$ holds for every structure $\mathfrak{A}$ for L, we write $\models\Phi$. Let Γ be a sentence or a set of sentences of L. A structure $\mathfrak{A}$ for L is called a model of Γ if $\mathfrak{A}\models\Gamma$ or $\mathfrak{A}\models\Phi$ for every Φ in Γ. We denote by $\mathcal{M}(\Gamma)$ the class of all models of Γ. If $\mathcal{M}(\Gamma)$ is not empty, we say that Γ is *satisfiable*. Furthermore, let Δ be a