# 7. *On a Property of Quadratic Farey Sequences*

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**§ 1. Introduction and notations.** The well known Farey sequence of order $s$ on $[0, 1]$ is in reality an ordered list of all zeros of linear polynomials $ax-b$ with integral coefficients satisfying $0 \leqq b \leqq a \leqq s$. The quadratic Farey sequence of order $s$ is defined as ordered list of all the real roots of the equation $ax^2+bx+c=0$, which $0 \leqq a \leqq s$, $|b| \leqq s$, $0 \neq |c| \leqq s$. Recently, H. Brown and K. Mahler study the quadratic Farey sequence on $[0, 1]$, and give some data via the computer [1]. In this paper, we give a formula to Table II, [1], i.e. the value of the determinant formed by the coefficients of three consecutive quadratics at certain rational points.

In this paper itallic letters and letters with a suffix or sign, $r_p^*$, $l_p$ etc. denote all integers except $x$, $y$. The symbol $[q/p]$ denotes the integral part of $q/p$; that is, the integer such that $[q/p] \leqq q/p < [q/p]+1$. Put

$$L_s=\{(a, k, l): s \geqq a \geqq 0,\ 0 \neq |l| \leqq s,\ |k| \leqq s\}$$
$$N_{s,r}^{+}=\{(l, k): nl-mk=r,\ 0<l \leqq s,\ |k| \leqq s\}$$
$$N_{s,r}^{-}=\{(l, k): nl-mk=r,\ 0>l \geqq -s,\ |k| \leqq s\}$$

$d(a, r, k, l)=d_{m/n}(a, r, k, l)=|(m/n, m/n)|-|$the point which (1) $y=l/(ax+k)$ intersects with (2) $y=x|$, where $|*|$ denotes the length of a vector $*$. Now we denote an order to the set $M_{s,r}$, where $M_{s,r}=N_{s,r}^{+}$ or $N_{s,r}^{-}$. If $M_{s,r} \neq \emptyset$, $(l, k)<(l', k') \Leftrightarrow |l|<|l'|$ and $(l, k)=(l', k') \Leftrightarrow l=l'$. Here we call $(l, k)$ or $l$ maximum in $M_{s,r}$ when the value $|l|$ is maximum among the element $(l, k) \in M_{s,r}$.

In order to obtain the results, we consider fractional functions (1) $y=l/(ax+k)$ for $(a, k, l) \in L_s$ and the equation (2) $y=x$. Then, the set $M_s$ of all the positive points on $[0, 1]$ which (1) is intersecting with (2) gives the quadratic Farey sequence of order $s$. The necessary and sufficient condition that (1) throws the point $(m/n, m/n)$ is $a=nr/m^2$, where $r=nl-mk$, but $a=nr/m^2$ is not necessary integral number, so, we must find the fractional function (1) with integral coefficients throwing the nearest point to $(m/n, m/n)$. That is, it is reduced to find two elements $(a, k, l) \in L_s$ such that $d(a, r, k, l)>0$ is minimum and $d(a, r, k, l)<0$ is maximum. Here we call the equation giving this nearest point smaller (larger) than $m/n$ lower (upper) best approximating equation with respect to $m/n$. Our results are given as Theorems 1–3.