6. A Note on Partially Hypoelliptic Operators

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1. Introduction. We shall study in this note the hypoellipticity of the following partial differential operator,

(1.1) $P(t; D_x, \partial_t) = \partial_t + ait^{l_0}D_x^m + bt^{l_1}D_x^{2n}, \qquad (a, b \in \boldsymbol{R}, i = \sqrt{-1}),$

where $\partial_t = \partial/\partial t$, $D_x = -i\partial/\partial x$ and $(x, t) \in \boldsymbol{R}_x \times (-1, 1)$.

Concerning hypoelliptic operators various studies have been made by many authors. One of the recent developments is that of degenerate operators. In this case almost studies are concentrated in the relation between the order of derivative and that of degeneracy of the coefficient, and there arise interesting properties which do not occur in the regular case. The difficulties lie on how to be dissolved the singularity appeared on a submanifold (or a subset) where the operator degenerates (see [1]~[9] and those references).

Contrary to this point of view, our purpose in this note is to show that under some conditions the operator (1.1) is regular (in some sense) on $t=0$, but is not regular on $t=t_0\neq 0$.

Let us now present an exact statement of our result. For this purpose we assume,

$$(1.2)\quad \begin{cases} \text{(i)} & m > 2n, \\ \text{(ii)} & l_0 \text{ and } l_1 \text{ are a non-negative integer and a non-negative} \\ & \text{even integer respectively,} \\ \text{(iii)} & a\cdot b \neq 0, \\ \text{(iv)} & (m-1)/(l_0+1) < 2n/(l_1+1). \end{cases}$$

Then we have

Theorem. *Under the assumptions* (1.2) *the operator given by* (1.1) *has the following properties*;

(i) *P and its adjoint ${}^tP$ are hypoelliptic on $t=0$ with respect to x, i.e., if $Pu \in C^\infty(I_x \times J_t)$ and $u \in \mathcal{E}^0(J_t; \mathcal{D}'(I_x))$, then $u(x, 0) \in C^\infty(I_x)$, where $I_x = (-\alpha, \alpha)$, $J_t = (-\beta, \beta)$. It also holds for ${}^tP$.*

(ii) *P and ${}^tP$ are not hypoelliptic on $t=t_0\neq 0$ with respect to x.*

Remark. (i) If m, l_0 and l_1 are even integers, $\mathrm{Re}\, ai > 0$ and $\mathrm{Re}\, b > 0$ (or if m and l_0 are even integers, $\mathrm{Re}\, ai > 0$ and $m/(l_0+1) \geqq 2n/(l_1+1)$), then P and ${}^tP$ are hypoelliptic in $\boldsymbol{R}_x \times (-1, 1)$.

(ii) If m is an even integer, l_0 and l_1 are odd integers, $\mathrm{Re}\, ai > 0$ and $\mathrm{Re}\, b > 0$ (or if m is an even integer, l_0 is an odd integer, $\mathrm{Re}\, ai > 0$