5. The Determinant of Matrices of Pseudo-differential Operators

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The purpose of this paper is to give a definition of the determinant of matrices of pseudo-differential operators (of finite order) and to establish some of its properties. Let X be a complex manifold, and P^*X (resp. T^*X) be its cotangent projective (resp. vector) bundle. The projection from T^*X-X onto P^*X is denoted by γ.

Our result is the following.

Theorem. *For every matrix* $A(x,D)=(A_{ij}(x,D))_{1\leq i,j\leq N}$, *whose entries* $A_{ij}(x,D)$ *are pseudo-differential operators defined on an open set* $U\subset P^*X$, *one can canonically associate* $\det A(x,D)$, *which is a homogeneous holomorphic function defined on* $\gamma^{-1}(U)$, *and possesses the following properties*

a) $\det A(x,D)B(x,D)=\det A(x,D)\cdot\det B(x,D)$

b) $\det(A(x,D)\oplus B(x,D))=\det A(x,D)\cdot\det B(x,D)$

c) *if there are integers* m_i *and* n_j *such that order* $A_{ij}(x,D)\leq m_i+n_j$ *and* $\det(\sigma_{m_i+n_j}(A(x,D))$ *does not vanish identically, then*

$$\det A(x,D)=\det(\sigma_{m_i+n_j}(A_{i,j})),$$

where $\sigma_{m_i+n_j}(A_{ij})$ *denotes the principal symbol of* A_{ij} (*which is* 0 *if* A_{ij} *is of the order* $\leq m_i+n_j-1$). *In particular, our determinant reduces to the concept of the principal symbol, if the size* N *is* 1.

d) $A(x,D)$ *is invertible if and only if* $\det A(x,D)$ *vanishes nowhere.*

e) *if* $P(x,D)$ *is a pseudo-differential operator such that* $[P,A]=0$, *then* $\{\sigma(P),\det A\}=0$.

Corollary. *If* $A(x,D)$ *is a matrix of differential operators, then* $\det A(x,D)$ *is a homogeneous polynomial on the fiber coordinate* ξ.

Corollary is an immediate consequence of Theorem. In fact, by adding an auxiliary parameter t, one can regard $A(x,D)$ as a pseudo-differential operator defined on a (t,x)-space $C\times X$. Therefore, $\det A(x,D)$ is defined all over T^*X, which implies $\det A(x,D)$ is a polynomial on ξ.

In order to prove Theorem, we prepare the following lemma.

Lemma (see [2]). *Let* K *be a* (*not necessarily commutative*) *field,* $K=\bigcup_{m\in Z}K_m$ *be a filtration of* K *satisfying*

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