4. *A Remark on the Rational Points of Abelian Varieties with Values in Cyclotomic Z_p-Extensions*

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Let K be an algebraic number field of finite degree, p a prime integer, L/K a Z_p-extension (or Γ-extension), and let A be an abelian variety defined over K. With these settings, recently Mazur [3] investigated the problem concerning the finite generatedness of the group of rational points $A(L)$. He obtained some sufficient conditions for affirmative solution of this problem. In this note we prove that the torsion part of $A(L)$ is finite if L/K is cyclotomic and if A has good reduction at some prime dividing p. In fact we prove a more general theorem:

Theorem. *Let K be a finite extension field of Q_p, L the smallest field containing K and all p-power roots of* 1, *and let A be an abelian variety defined over K which has good reduction. Then the torsion part of $A(L)$ is finite.*

Proof. First we show that there is a finite extension K'/K contained in L such that L/K' is a totally ramified extension. In fact, take a finite extension E/Q such that $E\otimes_Q Q_p=EQ_p=K$ (cf. Lang, Algebraic Number Theory, Chap. II, § 2, Proposition 4, Corollary). Let F be the smallest field containing E and all p-power roots of 1. From [1], § 7 and [3], § 2(c), there is a finite extension E'/E contained in F such that for some prime v of E' dividing p, F/E' is totally ramified at v. Then, putting K' to be the completion of E' at v, we obtain the desired field. From now on, taking K' instead of K, we assume that L/K is totally ramified. Now denote by $A(L)^{(p')}$ the p'-primary part of $A(L)$, and take $y\in A(L)^{(p')}$. If p' is relatively prime to p, then, by [8], Theorem 1, $K(y)/K$ is an unramified extension, and this means $y\in A(K)^{(p')}$. Hence $A(L)^{(p')}$ is contained in $A(K)^{(p')}$ and, from the well known fact that the torsion part of $A(K)$ is finite, we conclude that $A(L)^{(p')}$ is finite for all primes p' distinct from p and is zero for almost all p'. Therefore it is sufficient to consider the p-part $A(L)^{(p)}$.

We denote by $T_p(A)$ the Tate-module of A, $T_p(A(L))$ the fixed points of $T_p(A)$ under $\mathrm{Gal}(\bar{K}/L)$, where $\bar{K}$ is the algebraic closure of K. By the elementary divisor theorem, under suitable basis we can write these modules as: $T_p(A)=Z_p\oplus\cdots\oplus Z_p$, $T_p(A(L))=p^{a_1}Z_p\oplus\cdots\oplus p^{a_n}$