23. *The Fundamental Solution for a Parabolic Pseudo-Differential Operator and Parametrices for Degenerate Operators*

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Introduction. In the present paper we shall construct the fundamental solution $E(t,s)$ for a parabolic pseudo-differential equation

$$\begin{cases} Lu=\dfrac{\partial u}{\partial t}+p(t\,;x,D_x)u=0 & \text{in } (0,\infty)\times R^n \\ u|_{t=0}=u_0 \end{cases} \tag{0.1}$$

where $p(t\,;x,D_x)$ is a pseudo-differential operator of class $\mathcal{E}_t^0(S^m_{\lambda,\rho,\delta})$ $(0\leqslant\rho\leqslant 1,\ -\infty<\delta<1,\ \delta<\rho)$ which satisfies the following condition:

There exist positive constants C_0 and R such that

(0.2) $\operatorname{Re} p(t\,;x,\xi)\geqslant C_0\lambda(x,\xi)^m$ for $0\leqslant t<\infty$ and $|x|+|\xi|\geqslant R$,

where $\lambda=\lambda(x,\xi)$ is a basic weight function defined in § 1. We note that $\lambda(x,\xi)$ varies even in x and may increase in polynomial order, and that it is important to take $\delta<0$ in § 4.

The fundamental solution $E(t,s)$ will be constructed as a pseudo-differential operator of class $S^0_{\lambda,\rho,\delta}$ with parameter t and s. The method of construction of $E(t,s)$ is similar to that given in Tsutsumi [10]. Then the solution of the Cauchy problem (0.1) is given by $u(t)=E(t,0)u_0$.

In § 3 we show that if $P(t)$ is a positive operator, then $\exp\{c(t-s_0)E(t,s_0)\}$ are bounded in $S^{-N}_{\lambda,\rho,\delta}$ for $t\geqslant t_0>s_0\geqslant 0$, where c is a positive constant and N is any number.

As an application of the above theorems, in § 4 we construct the fundamental solution $E_0(t)$ for a degenerate parabolic operator

$$L_0=\frac{\partial}{\partial t}+D_x^{2l}+x^{2k}D_y^{2m}=\frac{\partial}{\partial t}+P_0 \tag{0.3}$$

and apply $E_0(t)$ to construct the parametrix for P_0 near $x=0$ in some class of pseudo-differential operator. We note that in case $l=k=m=1$ the precise symbol of the fundamental solution $E_0(t)$ is found in Hoel [4] and that the operator P_0 has been studied by Beals [1], Hörmander [3], Grushin [2], Kumano-go and Taniguchi [6] and Sjöstrand [9].

§ 1. **Notations and basic calculus of pseudo-differential operators of class $S^m_{\lambda,\rho,\delta}$.** We say that a C^∞-function $\lambda(x,\xi)$ in $R^n_x\times R^n_\xi$ is a basic weight function when $\lambda(x,\xi)$ satisfies conditions (cf. [6]):