32. On the Uniqueness of Global Generalized Solutions for the Equation $F(x, u, \operatorname{grad} u)=0$

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1. Introduction. If we intend to treat the Cauchy problem for the Hamilton-Jacobi equation

$$u_t+f(\operatorname{grad} u)=0, \qquad x \in R^n, \quad t>0,$$
$$(\operatorname{grad} u=(u_{x_1}, \cdots, u_{x_n}))$$

from the point of view of the theory of semigroups of nonlinear transformations, it is necessary ([1]) to establish the existence and uniqueness of certain bounded (possibly generalized) solutions of the associated equation

(AE) $$u+f(\operatorname{grad} u)=h(x), \qquad x \in R^n,$$

for given h. In this note we shall consider a more general equation of the form

(E) $$F(x, u, \operatorname{grad} u)=0, \qquad x \in R^n,$$

and prove a uniqueness theorem for certain bounded generalized (Lipschitz-continuous) solutions of (E). A semigroup treatment of the Hamilton-Jacobi equation in several space variables will be taken up in a later paper.

2. Definition of a generalized solution. We shall assume that the function $F(x, u, p)$ in (E) is real-valued and of class C^2 with respect to all its arguments in $R^n_x \times R^1_u \times R^n_p$ and satisfies the following three conditions:

i) The matrix $(F_{ij}(x, u, p))$, where $F_{ij}=\partial^2 F/\partial p_i \partial p_j$ $(i, j=1, \cdots, n)$, is nonnegative, i.e.,

$$\sum_{i,j=1}^{n} F_{ij}(x, u, p)\lambda_i \lambda_j \geq 0$$

for each $(x, u, p) \in R^n_x \times R^1_u \times R^n_p$ and each real $\lambda_1, \cdots, \lambda_n$;

ii) There exists a positive constant c such that

$$F_u(x, u, p) \geq c$$

for all $(x, u, p) \in R^n_x \times R^1_u \times R^n_p$;

iii) The partial derivatives $F_{p_i}, F_{p_i x_i}, F_{p_i u}$ and $F_{p_i p_i}$ $(i=1, \cdots, n)$ are bounded in any subdomain

(1) $$\mathcal{D}=\{(x, u, p)\,;\, x \in R^n, |u| \leq U, |p| \leq P\},$$

where U and P are arbitrary constants.

Under the assumption i), we shall give the following definition (cf. [3], [4]).