54. *On the Freudenthal's Construction of Exceptional Lie Algebras*

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Introduction. In his papers [3], [4], Professor Freudenthal constructed an exceptional simple Lie algebra $\mathfrak{G}$ as follows. Let $\mathfrak{J}$ be an exceptional simple Jordan algebra of all 3×3 Hermitian matrices with coefficients in the algebras of octaves, in which the Jordan product $X\cdot Y$ is defined as $1/2(XY+YX)$. A symmetric cross product $X\times Y$ in $\mathfrak{J}$ is defined by

$$X\times Y=X\cdot Y-\frac{1}{2}(\mathrm{sp}\,(X)Y+\mathrm{sp}\,(Y)X-\mathrm{sp}\,(X)\,\mathrm{sp}\,(Y)I+(X,Y)I),$$

where $\mathrm{sp}\,(X)$ means the spur of X, I is the unit matrix and $(X, Y)=\mathrm{sp}\,(X\cdot Y)$ for $X, Y\in\mathfrak{J}$. Furthermore, for any $X, Y\in\mathfrak{J}$, $\langle X, Y\rangle$ is a linear transformation of $\mathfrak{J}$ defined by

$$\langle X, Y\rangle Z=2Y\times(X\times Z)-\frac{1}{2}(Z, Y)X-\frac{1}{6}(X, Y)Z \qquad \text{for } Z\in\mathfrak{J}.$$

Let $\mathfrak{H}$ be the subspace spanned by $\{\langle X, Y\rangle \mid X, Y\in\mathfrak{J}\}$ in the space of linear transformations on $\mathfrak{J}$. Let $\mathfrak{K}=\mathfrak{J}\oplus\mathfrak{J}\oplus\boldsymbol{R}\oplus\boldsymbol{R}$ and $\mathfrak{L}=\mathfrak{H}\oplus\boldsymbol{R}\oplus\mathfrak{J}\oplus\mathfrak{J}$ ($\boldsymbol{R}$ is the field of real numbers) be direct sums as vector spaces, in which elements are denoted as

$$P=\ulcorner X, Y, \xi, \omega\urcorner \quad\text{and}\quad \Theta=\ulcorner \textstyle\sum_i\langle X_i, Y_i\rangle, \rho, A, B\urcorner$$

or

$$P=\begin{bmatrix}X\\Y\\\xi\\\omega\end{bmatrix}\quad\text{and}\quad \Theta=\begin{bmatrix}\sum_i\langle X_i, Y_i\rangle\\\rho\\A\\B\end{bmatrix}.$$

For any elements $P_i=\ulcorner X_i, Y_i, \xi_i, \omega_i\urcorner$ $(i=1,2)$ in $\mathfrak{K}$, an alternating form $\{P_1, P_2\}$ and an element $P_1\times P_2$ of $\mathfrak{L}$ are defined as follows;

$$\{P_1, P_2\}=(X_1, Y_2)-(X_2, Y_1)+\xi_1\omega_2-\xi_2\omega_1,$$

$$P_1\times P_2=\frac{1}{2}\begin{bmatrix}\langle X_1, Y_2\rangle+\langle X_2, Y_1\rangle\\-\frac{1}{4}((X_1, Y_2)+(X_2, Y_1)-3\xi_1\omega_2-3\xi_2\omega_1)\\-Y_1\times Y_2+\frac{1}{2}(\xi_1X_2+\xi_2X_1)\\X_1\times X_2-\frac{1}{2}(\omega_1Y_2+\omega_2Y_1)\end{bmatrix}.$$

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