# 52. *On Subclasses of Hyponormal Operators*

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**1.** We shall consider a (bounded linear) operator $T$ acting on a Hilbert space $\mathfrak{H}$. An operator $T$ is *hyponormal* if $TT^* \leqq T^*T$. And $T$ is *quasinormal* if $T$ commutes with $T^*T$. In [2] and [3], Campbell has discussed a subclass of hyponormal operators: An operator $T$ is *heminormal* if $T$ is hyponormal and $T^*T$ commutes with $TT^*$. The subclass is called $(BN)^+$ in [3]. Also he proved

**Theorem A.** *If $T$ is heminormal, then $T^n$ is hyponormal for every $n$.*

We shall define a new class of operators to improve Theorem A. For each $k$, an operator $T$ is *$k$-hyponormal* if

$$(1) \qquad (TT^*)^k \leqq (T^*T)^k.$$

Since $f(\lambda)=\lambda^\alpha$ for $0\leqq\alpha\leqq 1$ is operator monotone, every $k$-hyponormal operator is hyponormal.

In this note, in § 2 we shall give characterizations of heminormal, quasinormal and $k$-hyponormal operators by means of an operator equation due to Douglas [4]. In § 3, we shall show that every heminormal operator is $n$-hyponormal for every $n$, and for each $k$, if $T$ is $k$-hyponormal, then $T^k$ is hyponormal.

**2.** In this section, we shall characterize heminormal, quasinormal and $k$-hyponormal operators. In [4], Douglas showed the following

**Theorem B.** *Let $A$ and $B$ be operators on $\mathfrak{H}$. Then $AA^* \leqq \lambda^2 BB^*$ for some $\lambda \geqq 0$ if and only if there is an operator $C$ such that $A=BC$.*

In the proof of Theorem B, an operator $C$ is constructed as follows; (i) $C^*(B^*x)=A^*x$ for every $x \in \mathfrak{H}$, (ii) $C^*$ vanishes on $\operatorname{ran}(B^*)^\perp$, and (iii) $\|C\| \leqq \lambda$.

Now we shall give a characterization of heminormal operators.

**Theorem 1.** *An operator $T$ is heminormal if and only if there is a positive contraction $P$ such that*

$$(2) \qquad TT^* = PT^*T.$$

**Proof.** Suppose that $T$ is heminormal. Since $T^*T$ commutes with $TT^*$, we have $(TT^*)^2 \leqq (T^*T)^2$. It follows from Theorem B that there is an operator $C$ such that $TT^*=T^*TC$, i.e., $TT^*=C^*T^*T$. So we put $P=C^*$, then we have by the above remarks (i) and (ii)

$$(P(x_1+x_2), x_1+x_2) = (Px_1, x_1) \geqq 0$$

for every $x_1 \in \overline{\operatorname{ran}(T^*T)}$ and $x_2 \in \operatorname{ran}(T^*T)^\perp$, that is, $C^* \geqq 0$. Since $P$