50. *On a Problem of E. L. Stout*

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1. Introduction. The following very interesting theorem of T. Radó [3] was proved by many mathematicians (H. Behnke-K. Stein, H. Cartan, I. Glicksberg, M. Goldstein and T. R. Chow, E. Heinz, R. Kaufman and T. Radó, etc.).

Theorem of Radó. *Let $f(z)$ be a complex-valued continuous function defined in $\{|z|<1\}$. If $f(z)$ is analytic in each component of $\{|z|<1\}-f^{-1}(0)$, then $f(z)$ is analytic in $\{|z|<1\}$.*

On the other hand, E. L. Stout [5] proved the possibility of replacing the set $\{|z|<1\}-f^{-1}(0)$ by $\{|z|<1\}-f^{-1}(E)$ where E is a set of capacity zero. Moreover, he proposed another possibility of $\{|z|<1\}-f^{-1}(0)$ by $\{|z|<1\}-f^{-1}(E)$ where E is a set of positive capacity. In this paper, the present author will give an answer to this problem under some condition.

2. Notation and terminology. Let G be a $n+1$-ply connected region on an open Riemann surface R whose boundary consists of $n+1$ rectifiable closed analytic Jordan curves $C_0, C_1, \cdots, C_n$, where C_0 contains $C_1, \cdots, C_n$ in its interior. Let ω be the harmonic measure in G with boundary values 0 on C_0 and 1 on $C_1, \cdots, C_n$. We call $\mu=2\pi/D_G(\omega)$ the harmonic modulus of G where $D_G(\omega)$ is the Dirichlet integral of ω over G.

Proof of the Theorem. Lemma (Sario) (cf. [4]). *Let R be an open Riemann surface. If there exists a normal exhaustion $\{R_n\}$ satisfying $\sum_{n=1}^{\infty}\mu_n^*=\infty$, where μ_n^* is the minimum harmonic modulus of connected components of R_n-R_{n-1}, then R belongs to O_{AD}.*

We shall prove

Theorem. *Let U be an open unit disk $\{|z|<1\}$ and F be a compact set in the complex plane $\mathbf{C}$. Let $f(z)$ be a complex-valued continuous function on $\overline{U}$. Set $E=f^{-1}(F)$. Suppose f is analytic in each component of $\overline{U}-E$ and the valence function $n_f(w)$ is finite. If $\hat{C}-F$ belongs to O_{AD} in the sense of Sario ($\hat{C}$ is the one point compactification of $\mathbf{C}$), then the set E is of class N_D.*[1] *Moreover if $D_{U-E}(f)<\infty$, then f is analytic in $\overline{U}$ and $D_U(f)<\infty$.*

Proof. First, suppose $n_f(w)$ is bounded and $n_f(w)\leqq N_f$. Let $\{R_n\}$

1) See [1].