# 49. *Automorphic Forms and Algebraic Extensions of Number Fields*[*)]

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§ 0. The purpose of this paper is to present a result on an arithmetical relation between Hilbert cusp forms over a totally real algebraic number field, which is a cyclic extension of the rational number field $\boldsymbol{Q}$ with a prime degree $l$, and cusp forms of one variable. The details of this result will appear in [7].

Let $F$ be a totally real algebraic number field, and $\mathfrak{o}$ be its maximal order. For an even positive integer $\kappa$, let $S_\kappa(\Gamma)$ denote the space of Hilbert cusp forms of weight $\kappa$ with respect to the subgroup $\Gamma = GL_2(\mathfrak{o})^+$ consisting of all elements with totally positive determinants in $GL_2(\mathfrak{o})$. For a place (archimedean or non-archimedean) $v$ of $F$, let $F_v$ be the completion of $F$ at $v$. For a non-archimedean place $v$ $(=\mathfrak{p})$, let $\mathfrak{o}_\mathfrak{p}$ be the ring of $\mathfrak{p}$-adic integers of $F_v$. Let $F_A$ be the adele ring of $F$, and consider the adele group $GL_2(F_A)$. Let $\mathfrak{U}_F$ be the open subgroup $\prod_{\mathfrak{p}:\text{ non-archimedean}} GL_2(\mathfrak{o}_\mathfrak{p}) \times \prod_{v:\text{ archimedean}} GL_2(F_v)$ of $GL_2(F_A)$. Then we can consider the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$ and its action $\mathfrak{T}$ on $S_\kappa(\Gamma)$ as in G. Shimura [8].

For the ordinary modular group $SL_2(\boldsymbol{Z})$ $(=GL_2(\boldsymbol{Z})^+)$, we also consider its adelization $\mathfrak{U}_\boldsymbol{Q} = \prod_p GL_2(\boldsymbol{Z}_p) \times GL_2(\boldsymbol{R})$ and the Hecke ring $R(\mathfrak{U}_\boldsymbol{Q}, GL_2(\boldsymbol{Q}_A))$. The latter is acting on the space $S_\kappa(SL_2(\boldsymbol{Z}))$ of cusp forms of weight $\kappa$ with respect to $SL_2(\boldsymbol{Z})$.

§ 1. The space $\mathcal{S}_\kappa(\Gamma)$. Suppose $F$ is a cyclic extension of $\boldsymbol{Q}$ of degree $l$. We fix an embedding of $F$ into the real number field $\boldsymbol{R}$ and a generator $\sigma$ of the Galois group Gal $(F/\boldsymbol{Q})$ of the extension $F/\boldsymbol{Q}$, then all the distinct embeddings of $F$ into $\boldsymbol{R}$ are given by $\sigma^i$, $0 \leq i \leq l-1$. We consider the group $GL_2(F)$ as a subgroup of $GL_2(\boldsymbol{R})^l$ by $g \to (g, {}^\sigma g, \cdots, {}^{\sigma^{l-1}}g)$ for $g \in GL_2(F)$. For this fixed generator $\sigma$, we define an operator $T_\sigma$ on $S_\kappa(\Gamma)$ by the permutation of variables, namely $T_\sigma f(z_1, \cdots, z_l) = f(z_2, \cdots, z_l, z_1)$ for $f \in S_\kappa(\Gamma)$. Using this $T_\sigma$, we define a new subspace $\mathcal{S}_\kappa(\Gamma)$ of $S_\kappa(\Gamma)$, to be called "the space of symmetric Hilbert cusp forms", as follows;

$$\mathcal{S}_\kappa(\Gamma) = \{f \in S_\kappa(\Gamma) \mid \mathfrak{T}(e)T_\sigma f = T_\sigma \mathfrak{T}(e) f \quad \text{for any } e \in R(\mathfrak{U}_F, GL_2(F_A))\}.$$

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