118. *Roots of Operators*

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We can completely determine nth roots of operators on Banach spaces when their spectra suit our convenience. One of the cases is given in Theorem 1. It is extremely connected with results by E. Hille (Theorems 1, 2 and 3 in [2]), those by J. G. Stampfli (Lemma 1 and Theorem 1 in [4]) and those by M. R. Embry (Thorems 3 and 4 in [1]). An application is an observation of the structure of periodic bounded automorphisms of Banach algebras. It is sammalized in Theorem 2.

1. Throughout this paper, we mean by an operator a bounded linear operator; n denotes a positive integer and $Sp(S)$ the spectrum of an operator S.

Theorem 1. *Suppose that S is an operator on a Banach space and that there exists on the plane a curve C leading from the origin 0 to the point at infinity such that $Sp(S)\cap C=\emptyset$. Then $\{z^{1/n}: 0\neq z\in C\}\cup\{0\}$, a union of n curves with an only common point 0, divides the plane n sectorial domains $D_0, \cdots, D_{n-2}$ and D_{n-1}, and it follows that*

(a) *for each D_k, there corresponds a unique nth root R_k of S such that $Sp(R_k)\subset D_k$; it necessarily is in the norm-closed algebra of operators generated by S and the identity operator I; and*

(b) *if T is an nth root of S, then it is of the form*

$$T=\sum_{k=0}^{n-1} R_k E_k,$$

where $E_0, \cdots, E_{n-2}$ and E_{n-1} are mutually orthogonal idempotent operators with $\sum_{k=0}^{n-1} E_k=I$, each of which commutes with S and hence with every R_k.

A part of the following proof is devoted to give the form of nth roots of I. It is not new; in fact, known by Stampfli in [3], but the way employed here is alternative and somewhat elementary.

Proof. Denote by f a branch of nth root function on the plane slit along the curve C, valued in D_0; and Γ a rectifiable Jordan contour having no common points with C, oriented in a counterclockwise direction. Define operators $R_0, \cdots, R_{n-2}$ and R_{n-1} by

$$R_0=f(S)=-\frac{1}{2\pi i}\int_\Gamma f(z)(S-zI)^{-1}dz,$$

and

$$R_k=\zeta_k R_0, \qquad k=1, \cdots, n-1,$$