106. *Divisors on Meromorphic Function Fields*

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Consider the field $M(R)$ of meromorphic functions on an *open* Riemann surface R. Let

$$f(z)=\sum_{n=-\infty}^{\infty} b_n(z-a)^n$$

be the Laurent expansion of an $f\in M(R)^*=M(R)-\{0\}$ at a point $a\in R$ where we use the same notation for generic points of R and their local parameters. The divisor $\partial_f(a)$ of $f\in M(R)^*$ at $a\in R$ is defined by

$$\partial_f(a)=\inf\{n\,;\,b_n\neq 0\}.$$

We first fix an $f\in M(R)^*$ in $\partial_f(a)$ and consider it as a function of a on R. Extracting the essence of the point function $\partial_f(\cdot):R\to\boldsymbol{Z}$ (the integers) we call a mapping $\partial(\cdot):R\to\boldsymbol{Z}$ a *divisor on* R if the set $\{z\in R\,;\,\partial(z)\neq 0\}$ is isolated in R. Then we have

(I) The Weierstrass-Florack Theorem. *For any divisor $\partial(\cdot)$ on R there exists a unique (up to multiplications by zero free holomorphic functions) $f\in M(R)^*$ such that $\partial(\cdot)=\partial_f(\cdot)$ on R.*

We next fix a point $a\in R$ in $\partial_f(a)$ and consider it as a functional of f on $M(R)^*$. As an abstraction of the functional $\partial.(a):M(R)^*\to\boldsymbol{Z}$ we say that a mapping $\partial.:M(R)^*\to\boldsymbol{Z}$ is a *divisor on* $M(R)^*$ if the following four conditions are satisfied:

(α) $\partial_{M(R)^*}=\boldsymbol{Z}$;

(β) $\partial_{C^*}=\{0\}$;

(γ) $\partial_{f\cdot g}=\partial_f+\partial_g$;

(δ) $\partial_{f+g}\geq\min(\partial_f,\partial_g)$,

where $\boldsymbol{C}$ is the field of complex numbers and $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$. As a counter part to (I) we have

(II) The Iss'sa Theorem. *For any divisor $\partial.$ on $M(R)^*$ there exists a unique point $a\in R$ such that $\partial.=\partial.(a)$ on $M(R)^*$.*

The crucial part of the proof of Iss'sa [2] of the above theorem is to show that

$$(\,*\,)\qquad \partial_z\geq 0 \textit{ for any divisor } \partial. \textit{ on } M(\boldsymbol{C})^*.$$

Observe that $M(R)=\{f/g\,;\,f,g\in A(R),\ g\not\equiv 0\}$ where $A(R)$ is the ring of holomorphic functions on R. Hence

$$d_f=\inf\{|\partial_g|\,;\,g\in M(\boldsymbol{C})^*\circ f,\ \partial_g\neq 0\}$$

is an integer not less than 1 for any fixed $f\in A(R)^*=A(R)-\{0\}$ with $\partial_f\neq 0$ and any fixed divisor $\partial.$ on $M(R)^*$, and there in fact exists a