156. *On the Difference between r Consecutive Ordinates of the Zeros of the Riemann Zeta Function*

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§ 1. **Introduction.** Let γ_n be the n-th ordinate of the zeros of the Riemann zeta function $\zeta(s)$ satisfying $0<\gamma_n\leq\gamma_{n+1}$. Here we are concerned with the following problems.

(i) To estimate $S_{r,k}(T)=\frac{1}{N(T)}\sum_{T<\gamma_n\leq 2T} d(\gamma_n, r)^k$ for integral $k\geqslant 1$ and $r\geqslant 1$, where $N(T)$ is the number of the zeros of $\zeta(s)$ in $0<\text{Re}\, s<1$, $0<\text{Im}\, s\leqslant T$ as usual and $d(\gamma_n, r)$ is $(\gamma_{n+r}-\gamma_n)/r$.

(ii) To estimate the number $N_r\left(\frac{C}{\log T}, T\right)$ of γ_n in $T<\gamma_n\leqslant 2T$ satisfying $d(\gamma_n, r)\geqslant C/\log T$.

As to (i) we have shown in [1], [3] that

$$S_{1,2}(T)\ll(\log T)^{-2}.$$

On the other hand the following result is announced in Zentralblatt [4];

$$S_{1,2k+1}(T)\ll\frac{(2k)!\,2^{2k}(2k+1)(\log\log T)^k}{k!\,(\log T)^{2k+1}}$$

for integral $k=o(\log T)$. Here we shall prove the following

Theorem 1. *Let $T>T_0$. Then for k in $1\leqslant k\ll(T\log T)^{2/3}$ and r in $1\leqslant r\ll k^{3/2}$, we have*

$$S_{r,k}(T)\ll\frac{(Ak)^{3k^2/(2k+1)}\,(\log(3+k))^k r^{-2k^2/(2k+1)}}{(\log T)^k},$$

where A is some positive absolute constant.

As to (ii) we have shown in [1], [3] that

$$N_r\left(\frac{2\pi(1+a)}{\log T}, T\right)\gg N(T)\exp(-(\log\log C)^{1-\varepsilon})$$

for $C>C_0$, integral r less than $A(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$ and

$a=(A(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}-r)/(C+A(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}-r)$,

where A's above (and also in this paper) are some positive absolute constants and ε's are arbitrarily small positive numbers. Here we shall prove

Theorem 2. *For $T>T_0$, $C>C_0$ and r in $1\leqslant r\leqslant T\log T\,C^{-1}$, we have*

$$N_r\left(\frac{C}{\log T}, T\right)\ll N(T)\exp(-A(rC)^{2/3}(\log rC)^{-1/3}).$$

§ 2. Proof of Theorem.

2–1. To prove our theorem we use the following