

174. On a Density Theorem of Linnik

By Yoichi MOTOHASHI

Department of Mathematics, College of Science and Technology,
Nihon University, Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Dec. 12, 1975)

1. Let $N(\alpha, T, \chi)$ be the number of zeros of Dirichlet's L -function $L(s, \chi)$ in the region $\alpha \leq \sigma \leq 1$, $|t| \leq T$, $s = \sigma + it$. Then, recently Montgomery and Selberg [5] (see also [2; p. 40]) have obtained the estimates

$$\sum_{\chi \pmod{q}} N(\alpha, T, \chi) \ll_{\epsilon} (qT)^{(3+\epsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* N(\alpha, T, \chi) \ll_{\epsilon} (Q^5 T^3)^{(1+\epsilon)(1-\alpha)}.$$

In their proof a refinement of the large sieve inequality [2; Théorème 7A] as well as the remarkable identity of Lemma 1 below (both due to Selberg) play vital roles. Also the other novelty of their method lies in that they have dispensed with the Power Sum Method of Turán which was essential in the former results due to Fogels and Gallagher in this field.

The purpose of this note is to announce a further improvement:

Theorem 1. *If $4/5 \leq \alpha \leq 1$, then we have*

$$\sum_{\chi \pmod{q}} N(\alpha, T, \chi) \ll_{\epsilon} (q^2 T^3)^{(1+\epsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* N(\alpha, T, \chi) \ll_{\epsilon} (Q^4 T^3)^{(1+\epsilon)(1-\alpha)}.$$

As for the proof we mention only that this is deduced by combining the lemmas of the next paragraph with the zero-detecting devices of [4; pp. 104–110], and that we actually obtain a result in which the exponent $1 + \epsilon$ is replaced by $1/\alpha + \epsilon$ in the above, then it can be expressed as in our theorem by the recent work of Jutila [3].

In the proof of the above result (i.e. in Lemma 2 below) we have used the Pólya-Vinogradov theorem, and if we use Burgess' result instead of it, then we can prove

Theorem 2. *In the vicinity of $\alpha = 1$, we have*

$$\sum_{\chi \pmod{q}} N(\alpha, T, \chi) \ll_{\epsilon} (q^{9/8} T^6)^{(1+\epsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* N(\alpha, T, \chi) \ll_{\epsilon} (Q^{9/4} T^6)^{(1+\epsilon)(1-\alpha)}.$$

The proof of the results of this note will be published elsewhere.

2. Our main lemmas are as follows:

Lemma 1. *Let r be a square-free integer and let $\phi_r(n) = \mu((r, n))\varphi((r, n))$. Further let*