169. *Approximation Theorem on Stochastic Stability*

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§ 1. This paper treats the approximation theorem on the stability theory of dynamical systems given by stochastic differential equations. Consider a dynamical system in $\boldsymbol{R}^n$:

$$(1)\qquad dx_i(t)=\sum_{k=1}^{n}\sigma_{ik}(x(t))dB_k(t)+b_i(x(t))dt \qquad (i=1,\cdots,n)$$

(in this paper, we always assume that coefficients of (1) are Lipschitz continuous). If we assume that for $m\geqq 1$

$$(2)\qquad \begin{cases}\sigma_{ik}(x)=\tilde{\sigma}_{ik}(\lambda)\,|x|^m+o(|x|^m)\\ b_i(x)=\tilde{b}_i(\lambda)\,|x|^{2m-1}+o(|x|^{2m-1}) \qquad |x|\to 0,\end{cases}$$

where $\lambda=x/|x|$, then the first approximation of (1) is defined by

$$(3)\qquad dx_i(t)=\sum_k \tilde{\sigma}_{ik}(\lambda(t))\,|x(t)|^m\,dB_k(t)+\tilde{b}_i(\lambda(t))\,|x(t)|^{2m-1}\,dt.$$

Following to Khas'minskii [2], we call $x(t)$ asymptotic stable in probability if $\lim_{|x|\to 0} P_x\{\lim_{t\to\infty}|x(t)|=0\}=1$, asymptotic unstable in probability if $P_x\{\lim_{t\to\infty}|x(t)|=\infty\}=1$ for all x $(\neq\{0\})$, divergent in probability if $P_x\{\sup_{t>0}|x(t)|>\varepsilon\}=1$ for all x $(\neq\{0\})$ and small $\varepsilon>0$.

The main theorems are:

Theorem 1. *If the solution of* (3) *is asymptotic stable in probability, then that of* (1) *is so.*

Theorem 2. *If the solution of* (3) *is asymptotic unstable in probability, then that of* (1) *is divergent in probability.*

When $m=1$, the results have been already proved by Khas'minskii [2] and Pinsky [4].

In § 2 we sketch proofs of Theorems 1 and 2. In § 3 they are applied to a limit behaviour of a stochastic process on a two dimensional compact manifold, which is useful for studying the stability of three dimensional linear systems (see [1]).

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§ 2. **Remark 1.** In this section it will be proved that the stability of (3) is equivalent to that of

$$(4)\qquad dx_i(t)=\sum_k \tilde{\sigma}_{ik}(\lambda(t))\,|x(t)|\,dB_k(t)+\tilde{b}_i(\lambda(t))\,|x(t)|\,dt.$$

Thus, a little modification of Khas'minskii's sharp stability criterion formulated in [1] is applicable to (3).