16. *An Interpolation of Operators in the Martingale H_p-spaces*

By Masami OKADA
Mathematical Institute, Tôhoku University

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1. Introduction. In this note we show that the Marcinkiewicz interpolation theorem of operators can be extended in the martingale setting.

2. Definition. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^{\infty})$ a probability space furnished with a non-decreasing sequence of σ-algebras of measurable subsets $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots \subset \mathcal{F} = \bigvee_{n=1}^{\infty} \mathcal{F}_n$.

We define the set of random variables $H_p = H_p(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^{\infty}) = \left\{ f \in L^p(\Omega)\,;\, |||f|||_p = \left[\int_\Omega (f^*)^p dP\right]^{1/p} < \infty \right\}$, where $f^*(w) = \sup_{1 \leqq n < \infty} |f_n(w)|$ and $p \geqq 1$.

Note that $H_1 \subsetneqq L^1$, and that $H_p = L^p$ for $1 < p < \infty$. In fact, there exists a constant A_p such that $\|f\|_p \leqq |||f|||_p \leqq A_p \|f\|_p$. Furthermore, as is well-known, the norm $|||f|||_p$ is equivalent to $\|(\sum_{n=1}^{\infty} |\Delta f_n|^2)^{1/2}\|_p$, where $\Delta f_n = f_n - f_{n-1}$, $f_0 = 0$ ([1]–[3]).

3. Weak type result. Let T be an operator from H_p to the set of random variables defined on a σ-finite measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Theorem. *Suppose that*

(1) *T is quasi-linear, i.e.* $|T(f+g)| \leqq C(|Tf| + |Tg|)$

(2) $\tilde{P}(\{w\,;\,|Tf(w)| > t\})^{1/q_i} \leqq M_i/t\, |||f|||_{p_i}$, *for all* $t > 0$, *where* $1 \leqq p_i \leqq q_i < \infty$ $(i = 0, 1)$, $p_0 \neq p_1$ *and* $q_0 \neq q_1$. *Let us put* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$, $0 < \theta < 1$. *Then*

$$\|Tf\|_q \leqq AC(C+1) M_0^{1-\theta} M_1^{\theta} \|f\|_p,$$

where

$$A^q = 0((q_1 - q)^{-1} + (q - q_0)^{-1}(p-1)^{-q_0}).$$

Proof. We consider the case $1 = p_0 < p_1$ and $q_0 < q_1$ only, the other cases are treated similarly.

1-st step. The following decomposition lemma is used in the proof, which corresponds to the Calderón-Zygmund decomposition ([4]–[6]).

Lemma (R. Gundy). *Let $v \in L^1(\Omega)$, $r \geqq 1$. Then for each $a > 0$, v is decomposed into three random variables g, h, k, $v = g + h + k$, which satisfy*

(1) $\quad P(\{w\,;\,g^*(w) > 0\}) \leqq K/a \|v\|_1, \qquad \|g\|_r \leqq K \|v\|_r$

(2) $\quad \left\| \sum_{n=1}^{\infty} |\Delta h_n| \right\|_1 \leqq K \|v\|_1, \qquad \|h\|_1 \leqq K \|v\|_1$