31. A Local Study of Some Additive Problems in the Theory of Numbers

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In this note we shall give some supplemental remarks to the author's previous work [5] on Goldbach's problem. As an application we shall solve some analogues of Titchmarsh divisor problem. The details of the latter will appear elsewhere.

We start with stating the following theorem which plays an essential role in our argument and whose slightly weaker form is proved and used in [5].

Theorem 1. *Suppose that* $\sum_{m\leqslant x}|b(m)|^2 \ll x(\log x)^C$ *with some positive absolute constant* C. *Then for any positive constants* A *and* b (<1), *there exists a positive constant* B *such that*

$$\sum_{d\leqslant Q}\operatorname*{Max}_{(a,d)=1}\left|\sum_{\substack{1\leqslant m\leqslant x^\delta\\(m,d)=1}} b(m)\left(\sum_{\substack{p\leqslant x/m\\p\equiv am^*(\mathrm{mod}\ d)}} 1-\frac{Li(x/m)}{\varphi(d)}\right)\right| \ll x(\log x)^{-A}$$

uniformly for δ *in* $0\leqslant\delta<1-(\log x)^{-b}$, *where* $Q=x^{1/2}(\log x)^{-B}$, $mm^*\equiv 1$ $(\mathrm{mod}\ d)$, $Li(x)=\int_2^x(\log x)^{-1}dx+0(1)$ *and* p *runs over primes.*

In fact, this is a generalization of Bombieri's mean value theorem (namely, $\delta=0$ and $b(1)=1$, Cf. [2] and [6]) and Chen's argument in [3]. We can also prove the following inequality under $b(m)\ll x^{1-\delta-\beta}$ for $m\leqslant x^\delta$, $\beta=(\log x)^{-f}$ with some f in $b<f<1$ in addition to the same circumstance as above;

$$\sum_{d\leqslant Q}\operatorname*{Max}_{(a,d)=1}\operatorname*{Max}_{1\leqslant y\leqslant x}\left|\sum_{\substack{mp\leq y,m\leqslant x^\delta\\mp\equiv a(\mathrm{mod}\ d)}} b(m)-\frac{1}{\varphi(d)}\sum_{\substack{mp\leqslant y\\m\leqslant x^\delta}} b(m)\right| \ll x(\log x)^{-A}.$$

We call this Theorem 1′. The conclusion in Theorem 1 (similarly for 1′) holds even if we replace $\sum_{\substack{p\leqslant x/m\\p\equiv am^*(\mathrm{mod}\ d)}} 1-\frac{Li(x/m)}{\varphi(d)}$ by $\sum_{\substack{p\leqslant x^{1-\delta}\\p\equiv am^*(\mathrm{mod}\ d)}} 1 -\frac{Lix^{1-\delta}}{\varphi(d)}$. We denote this by Theorem 1″. When our conclusion holds for any positive A and $Q=x^{1-\varepsilon}$ with any positive ε, we call it the generalized Halberstam-Richert's conjecture (G.H.R.).

Now let N be a sufficiently large even integer. Let $G(N)$ be the number of primes $p\leqslant N$ such that $N-p$ is a prime. Then

$$G(N)=P_N(N,N^{1/\alpha})-M_1(\alpha)-M_2(\alpha,2)+0(N^{1/\alpha})$$