29. *The Embedding Problem for Operator Groups*

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By a *semigroup* in a Banach space X we mean a one-parameter family $\{T_t: t\geqq 0\}$ of bounded linear operators on X such that (s_1) $T_0=I$ (the identity operator on X), $T_{t+s}=T_tT_s$ for $t, s\geqq 0$, and (s_2) for $x \in X$, T_tx is strongly measurable for $t>0$. A one-parameter family $\{G_t: t\in R\}$ of bounded linear operators on X is said to to be a *one-parameter strongly continuous group* in X, if (g_1) $G_0=I, G_{t+s}=G_tG_s$ for $t, s\in R$, and (g_2) for $x\in X, G_tx$ is strongly continuous on R with respect to t. Let $\{T_t\}$ be a semigroup in X. We say that the semigroup $\{T_t\}$ can be *embedded* in a group iff there exists a one-parameter strongly continuous group $\{G_t\}$ in X such that $G_t=T_t$ for $t\geqq 0$. A well-known theorem of Hille and Phillips ([1], Theorem 16.3.6.) states that a semigroup $\{T_t\}$ in X can be embedded in a group iff T_{t_0} is injective and surjective for some $t_0>0$.*) Our purpose in this paper is to give another version of this theorem in terms of Fredholm operator theory.

Let X and Y be Banach spaces. $B(X, Y)$ will denote the set of all bounded linear operators from X to Y. For basic properties of Fredholm operators, we refer to Schechter [2]. An operator $T\in B(X, Y)$ is said to be *Fredholm* if (f_1) $\alpha(T)\equiv\dim N(T)<\infty$, (f_2) $R(T)$ is closed, and $(f_3)\beta(T)\equiv\dim N(T^*)<\infty$, where $N(T)$, $R(T)$ and T^* denote the null space, the range and the adjoint operator of T, respectively. We denote by $\Phi(X, Y)$ the class of all Fredholm operators from X to Y. For $T\in\Phi(X, Y)$ we define the index $i(T)$ of T by $i(T)=\alpha(T)-\beta(T)$. We shall use the following facts concerning Fredholm operators:

(a) If $T_1\in\Phi(X, Y)$ and $T_2\in\Phi(Y, Z)$, then $T_2T_1\in\Phi(X, Z)$ and $i(T_2T_1)=i(T_1)+i(T_2)$.

(b) Assume that $T_1\in B(X, Y)$ and $T_2\in B(Y, Z)$ are such that $T_2T_1\in\Phi(X, Z)$. If either $\alpha(T_2)<\infty$ or $\beta(T_1)<\infty$, then $T_1\in\Phi(X, Y)$ and $T_2\in\Phi(Y, Z)$.

We now state our theorem:

Theorem. *A semigroup* $\{T_t\}$ *in* X *can be embedded in a group iff*

(E_1) $\bigcap_{t>0} N(T_t)=\{0\}$; *and*

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*) In [1] the semigroup $\{T_t\}$ is supposed to be of class (A), although it is proved without this assumption that the invertibility of some T_{t_0} implies that of every T_t; hence the theorem holds for every semigroup in X.