27. A Remark on the Character Rings of Finite Groups

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Introduction. The integral group ring ZG of a finite abelian group G is an important example of Gorenstein ring of dimension one (see [1], [2]). In this case, since ZG is isomorphic to the character ring R_G of G, we say that R_G is a Gorenstein ring. In this paper we show that the character rings of arbitrary finite groups are Gorenstein rings.

1. Let G be a finite group. Then the character ring R_G of G is a commutative ring and a finitely generated free Z-module. Its unity element is the principal character of G. As for group rings ([3]), we see that *R_G is isomorphic to the dual* $\mathrm{Hom}_Z(R_G, Z)$ *as R_G-modules.* This is equivalent to the existence of a nondegenerate symmetric bilinear form $(\ ,\ ): R_G \times R_G \to Z$ which satisfies the following conditions:

1) $(rs, t) = (r, st)$ for $r, s, t \in R_G$.

2) For each $f \in \mathrm{Hom}_Z(R_G, Z)$, there exists an $s \in R_G$ such that $f(r) = (r, s)$ for $r \in R_G$.

Such a bilinear form $(\ ,\ )$ is given by

$$(r, s) = \langle \bar{r}, s \rangle$$

for $r, s \in R_G$, using the ordinary inner product

$$\langle \mu, \nu \rangle = \frac{1}{|G|} \sum_{x \in G} \bar{\mu}(x) \nu(x),$$

where $\bar{\mu}$ denotes the function defined by $\bar{\mu}(x) = \mu(x^{-1})$ for $x \in G$. In fact, if $(r, s) = 0$ for all $r \in R_G$, then $\langle \chi, s \rangle = 0$ for all irreducible characters χ of G. Hence $s = 0$, which shows that $(\ ,\ )$ is nondegenerate. Moreover, for each $f \in \mathrm{Hom}_Z(R_G, Z)$, put

$$s = \sum_{\chi} f(\bar{\chi}) \chi,$$

where the sum is taken over all χ. Then $f(\chi) = (\chi, s)$ for all χ. Since $\{\chi\}$ is a Z-basis of R_G, we have $f(r) = (r, s)$ for all $r \in R_G$.

Hence R_G is a Frobenius Z-algebra in the sense of the definition given in [3]. It follows from Corollary 8 of [3] that R_G has a finite injective dimension. Thus from the fundamental theorem of [2] we obtain