# 48. On Symmetric Structure of a Group

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**1. Introduction.** Let $A$ be a set and $S$ a mapping of $A$ into the symmetric group on $A$. Denote the image of $a(\in A)$ under $S$ by $S_a$ or $S[a]$ and the image of $x(\in A)$ under $S_a$ by $xS_a$. Then $S$ is called a *symmetric structure* of $A$ if the following conditions are satisfied:

(i) $aS_a=a$, (ii) $S_a^2=I$ (the identity), (iii) $S[bS_a]=S_aS_bS_a$. A set with a symmetric structure is called a *symmetric set*. A symmetric set $A$ is called *effective* if $a\neq b$ implies $S_a\neq S_b$. Then group generated by $\{S_aS_b \mid a, b \in A\}$ is called the *group of displacements* and is denoted by $G(A)$. A symmetric structure of a finite set has been studied in [1] and [2].

Now let $A$ be a group. Then $A$ has symmetric structure $S$ defined by $xS_a=ax^{-1}a$. The purpose of this note is to study the structure of $G(A)$ for a given group $A$, and we shall determine it when the center $Z(A)$ of $A$ is trivial.

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**2. Group of displacements.** In this section we assume that $A$ is a group and $S$ is a symmetric structure of $A$ defined above.

**Proposition 1.** *$A$ is effective if and only if there is no involution in the center of $A$.*

**Proof.** Let $Z(A)$ be the center of $A$, and assume that $Z(A)$ contains an involution $t$. Then $xS_{at}=(at)x^{-1}(at)=ax^{-1}a=xS_a$. Therefore $A$ is not effective.

Conversely, assume that $A$ is not effective, then there exist distinct two elements $a$ and $b$ in $A$ such that $S_a=S_b$. Therefore, for any element $x$ in $A$,

(1) $$ax^{-1}a=bx^{-1}b.$$

Replacing $x$ with $e$ (the unit element) and $a$, we have

(2) $$a^2=b^2$$

(3) $$a=ba^{-1}b.$$

Then $b^{-1}a=(ab^{-1})^{-1}$ by (2), $(ab^{-1})^2=e$ by (3) and $(b^{-1}a)x^{-1}(ab^{-1})=x^{-1}$ for any $x$ in $A$. Hence, $ab^{-1}\in Z(A)$ and $(ab^{-1})^2=e$. Thus $Z(A)$ contains an involution.

Let $L_a$ and $R_a$ be permutations on $A$ such that

$$L_a: x\to ax,$$