47. *An Alternate Proof of a Transfer Theorem without using Transfer*

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(Comm. by Kenjiro SHODA, M. J. A., April 12, 1976)

In the paper [1] by the same author, he proved

Theorem A. *If a Sylow p-subgroup P of a finite group G has no quotient group isomorphic to the wreath product $Z_p \wr Z_p$, where Z_p is the cyclic group of order p, then $P \cap G' = P \cap N_G(P)'$.*

The purpose of this paper is to give a primitive proof of a particular case of this theorem. Namely, we shall prove

Theorem B. *If a Sylow 2-subgroup P of a finite group G has no quotient group isomorphic to the dihedral group D_8 of order 8, then $P \cap G^2G' = P \cap N^2N'$, where $N = N_G(P)$. In particular, if G has no subgroup of index 2, then so does N.*

Most of the notation is standard. Let G be a finite group. Then G' denotes the commutator group of G. For $X \subseteq G$, $\langle X \rangle$ is the subgroup generated by X. We set $G^2G' = \langle g^2, G' \mid g \in G \rangle$. We write $H \lhd G$ if H is a normal subgroup of G. For subgroups H, K of G, the notation $K \backslash H$ denotes the set $\{Kh \mid h \in H\}$. Clearly, every element of H induces a permutation on $K \backslash H$. We write $H < G$ if H is a proper subgroup of G.

The following lemma is essential to the proof of Theorem B.

Lemma. *Let P be a 2-group, $K < S < P$ and $x \in P$. Assume the following:*

(a) $|S : K| = 2$;

(b) *For any $u \in P$, $\langle x^2 \rangle^u \cap S \subseteq K$;*

(c) *The element x acts on the set $K \backslash P$ as an odd permutation.*

Then P has a quotient group isomorphic to D_8.

Proof. We shall argue by induction on $|P : S|$. Let R be a subgroup of P such that $|R : S| = 2$. Suppose $K \lhd R$. Since x acts on $K \backslash P$ as an odd permutation, we have that there is $u \in P$ such that x acts as an odd permutation on the set $K \backslash Ru\langle x \rangle$. Replacing x with uxu^{-1}, we may assume that $u = 1$. If x fixes an element of $K \backslash R\langle x \rangle$, then x acts trivially on $K \backslash R\langle x \rangle$, as $K \lhd R$, a contradiction. Thus x acts semiregularly on $K \backslash R\langle x \rangle$, and so the number of the $\langle x \rangle$-orbits of $K \backslash R\langle x \rangle$ is 1 or 3. It follows easily from $K \lhd R$ that $K \backslash R\langle x \rangle = K \backslash K\langle x \rangle$. Thus $|\langle x \rangle \cap R : \langle x \rangle \cap K| = 4$. This means that $x^j \in S - K$ for some even j. This contradicts the assumption of this lemma. Hence we proved