78. On the Summability of Taylor Series of the Regular Function of Bounded Type in the Unit Circle

By Chuji TANAKA
Mathematical Institute, Waseda University, Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., June 8, 1976)

1. Introduction. The object of this note is to introduce a new summation process, by means of which Taylor series of the regular function of bounded type in $|z|<1$ can be summable on $|z|=1$. The details of proofs will be published elsewhere in near future.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be the regular function of bounded type in the unit circle. In general, the series $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is not Cesàro-summable. In fact, put

$$f(z)=\exp\left(\frac{\alpha}{2}\cdot\frac{1+z}{1-z}\right)=\sum_{n=0}^{\infty} a_n z^n \qquad \text{for } \alpha>0,\ |z|<1,$$

which is the regular function of bounded type in the unit circle. Then we have

$$a_n=\exp\left(2\sqrt{\alpha n}+O(\ln n)\right)$$

([1] pp. 107–108). Since there exists no $k>-1$ such that $a_n=o(n^k)$, the series $\sum_{n=0}^{\infty} a_n z^n$ is not Cesàro-summable on $|z|=1$ (2 p. 78).

2. Notations and definitions. As usual, for $k>-1$, $|x|<1$, we put

$$\frac{1}{(1-x)^{k+1}}=\sum_{n=0}^{\infty} A_n^{(k)}\cdot x^n, \qquad \frac{1}{(1-x)^{k+1}}\cdot\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} S_n^{(k)}\cdot x^n,$$

where $S_n^{(k)}=\sum_{i=0}^{n} a_i A_{n-i}^{(k)}$, $C_n^{(k)}=S_n^{(k)}/A_n^{(k)}$. If $C_n^{(k)}\to s$ as $n\to\infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is Cesàro-summable (C,k) to s. For brevity, we write

$$\sum_{n=0}^{\infty} a_n=s(C,k).$$

Generalizing this Cesàro-summation, we introduce following summation process. For $k>-1$, $\alpha>0$ and $|x|<1$, we put

$$\frac{1}{(1-x)^{k+1}}\cdot\exp\left(\frac{\alpha}{1-x}\right)=\sum_{n=0}^{\infty} b_n(k,\alpha)\cdot x^n,$$

$$\frac{1}{(1-x)^{k+1}}\cdot\exp\left(\frac{\alpha}{1-x}\right)\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} S_n(k,\alpha)x^n,$$

where $S_n(k,\alpha)=\sum_{i=0}^{n} a_i b_{n-i}(k,\alpha)$, $C_n(k,\alpha)=S_n(k,\alpha)/b_n(k,\alpha)$. *If $C_n(k,\alpha)\to s$ as $n\to\infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is summable (C,k,α) to s.* For brevity, we write