98. On Discontinuous Groups Acting on a Real Hyperbolic Space. I

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1. This note gives a necessary and sufficient condition for a polyhedron in an n-dimensional real hyperbolic space to be a fundamental domain of some discontinuous group which has been established by B. Maskit [1] in 3-dimensional case. In 3-dimensional case our condition improves his parabolicity one and is much more combinatorial (See Definition 2). It also gives informations as to fixed point groups (See Theorem 2) analogous to the case of Coxeter groups ([2] Chapter IV).

We fix our notations. Let $\boldsymbol{H}$ be the subspace $\{\xi\in \boldsymbol{R}^n \mid \xi_1^2+\cdots+\xi_n^2<1\}$ of $\boldsymbol{R}^n$ with the metric form $ds^2=4\sum_{j=1}^n d\xi_j^2/(1-\sum_{j=1}^n \xi_j^2)^2$ $(n\geqq 2)$, which is called an n-dimensional real hyperbolic space. Let G be the group of all isometries of $\boldsymbol{H}$. Let F be an n-dimensional open polyhedron with totally geodesic faces in $\boldsymbol{H}$ satisfying the conditions: i) the number of faces is finite, ii) $\overline{\overline{F}}\cap\partial\boldsymbol{H}$ is a finite set, where $\overline{\overline{F}}$ means the closure of F and $\partial\boldsymbol{H}$ the boundary of $\boldsymbol{H}$ both under the topology of $\boldsymbol{R}^n$.

Some concrete examples will be given in the part II.

2. We define two kinds of "fitness" for F as follows.

Definition 1. A discrete subgroup Γ of G is said to be *fit* for F if the following conditions are satisfied: i) $\bigcup_{\gamma\in\Gamma}\gamma\overline{F}=\boldsymbol{H}$ where $\overline{F}$ means the closure of F under the topology of $\boldsymbol{H}$; for any element γ in Γ which is not the unit element we have $F\cap\gamma F=\emptyset$ and the family of the subsets $\{\gamma\overline{F}\}_{\gamma\in\Gamma}$ in $\boldsymbol{H}$ is locally finite, ii) Γ has no reflection and iii) the subset $\{\gamma\in\Gamma \mid \overline{F}\cap\gamma\overline{F}=\emptyset\}$ of Γ consists of finite elements.

Definition 2. Let $\mathcal{P}$ be a subdivision of $\overline{F}$ and $\mathcal{A}=\{\gamma_1,\cdots,\gamma_a\}$ be a subset of G. A pair $(\mathcal{P},\mathcal{A})$ is said to be *fit for* F if the following conditions are satisfied:

i) (*Structure of cell complex for* $\overline{F}$). $\mathcal{P}$ consists of a finite number of polyhedra each of which, called *facet*, is open in its support, namely the minimal subspace of $\boldsymbol{H}$ containing this facet. F is an element of $\mathcal{P}$. For any $F'\in\mathcal{P}$, $\overline{F'}$ is equal to the sum of $F''\in\mathcal{P}$ such that $F''\subset\overline{F'}$.

ii) (*Compatibility condition for* $\mathcal{P}$ *and* $\mathcal{A}$). Let $\mathcal{P}^{(\nu)}$ be the set of all $F'\in\mathcal{P}$ such that F' is ν-dimensional $(0\leqq\nu\leqq n)$. First $\mathcal{P}^{(n-1)}$ consists of even number of faces $\{H_1^+,H_1^-,\cdots,H_a^+,H_a^-\}$ such that $\gamma_i H_i^+=H_i^-$ and $\gamma_i F\cap F=\emptyset$ $(1\leqq i\leqq a)$. Secondly for any $F'\in\mathcal{P}$ we have $\gamma_i F'\in\mathcal{P}$ when ever $F'\subset H_i^+$. We say that a facet F' is *linked with* F'' by γ_i^ε if