97. *On Kronecker Limit Formula for Real Quadratic Fields*

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1. Let F be the real quadratic field with discriminant d embedded in the real field $\boldsymbol{R}$. Let χ be a primitive character of the group of the ideal class group modulo $\mathfrak{f}$ of F. *Assume that for a principal integral ideal* (μ) *of* F, $\chi((\mu))$ *is given by the following formula* (1).

(1) $$\chi((\mu))=\operatorname{sgn}(\mu)\chi_0(\mu),$$

where χ_0 is a character of the group of residue classes modulo $\mathfrak{f}$. Let $L_F(s,\chi)$ be the Hecke L-function of F associated with the character χ. In this note, we present a formula for the value $L_F(1,\chi)$ which seems to be new and suggestive. For previously known relevant results, we refer to E. Hecke [1], [2], G. Herglotz [3], C. Meyer [4], C. L. Siegel [6] and D. Zagier [7].

2. For a pair of positive numbers $a=(a_1,a_2)$, set

$$\begin{aligned} c_1(a)=&\frac{1}{a_1}\sum_{n=1}^{\infty}\left\{\psi\left(\frac{na_2}{a_1}\right)-\log\left(\frac{na_2}{a_1}\right)+\frac{a_1}{2na_2}\right\}\\ &+\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\log a_1-\frac{1}{2a_1}(\gamma-\log 2\pi)\\ &+\frac{a_1-a_2}{2a_1a_2}\log\frac{a_2}{a_1}-\frac{\gamma}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) \end{aligned}$$

and set

$$c_2(a)=\frac{1}{2a_1^2}\sum_{n=1}^{\infty}\left\{\psi'\left(\frac{na_2}{a_1}\right)-\frac{a_1}{na_2}\right\}+\frac{\pi^2}{12a_1^2}-\frac{1}{2a_1a_2}\log a_2+\frac{\gamma}{2a_1a_2},$$

where γ is the Euler constant and ψ is the logarithmic derivative of the gamma function.

Denote by $F(a,z)$ an entire function of z given by the following:

$$\begin{aligned} F(a,z)=z\exp\{-c_1(a)z-c_2(a)z^2\}\Pi'\left(1+\frac{z}{na_1+ma_2}\right)\\ \times\exp\left\{-\frac{z}{na_1+ma_2}+\frac{z^2}{2(na_1+ma_2)}\right\} \end{aligned}$$

where the product is over all pairs (n,m) of *non-negative integers* which are not simultaneously equal to zero.

We note that the function $F(a,z)^{-1}$ is the *double gamma function* introduced and studied by Barnes in [8].

Let $\varepsilon>1$ be the generator of the group of totally positive units of F. Choose a complete set of representatives $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_{h_0}$ of the group of *narrow* ideal classes of F. For each k $(1\leq k\leq h_0)$ set