116. A Note on Quasi Metric Spaces

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(Communicated by Kenjiro SHODA, M. J. A., Oct. 12, 1976)

1. Introduction and notations.

The purpose of this note is to point out errors in a proof and a theorem of Kim [3], and to give a corrected version of the theorem. By a quasi-metric on a set X we mean a non-negative real valued function p on $X \times X$ such that for $x, y, z \in X$ we have $p(x, y)=0$ if and only if $x=y$ and $p(x, y) \leq p(x, z)+p(z, y)$. The set $B(x, p, \varepsilon)=\{y \in X: p(x, y)<\varepsilon\}$ is the p-ball centre x and radius ε. The topology induced on X by p has the family $\{B(x, p, \varepsilon): x \in X, \varepsilon>0\}$ as a base. If p is a quasi-metric on X, its conjugate quasi-metric q on X is given by $q(x, y)=p(y, x)$ for $x, y \in X$. Bitopological concepts which are not defined are taken from Kelly [2].

2. A theorem and an example.

The following result is hinted at by Stoltenberg [6], and proved explicitly in [4].

Theorem 1. *Any quasi metric space whose conjugate quasi metric topology is compact is metrizable.*

Proof. Let T_1 be the topology induced on the set X by the quasi metric p whose conjugate q induces the compact topology T_2 on X. Let U be T_2 open, and $y \in U$. Since (X, T_1, T_2) is pairwise Hausdorff [2], for each $x \in X-U$ there is a T_2 open set U_x and a T_1 open set V_x such that $x \in U_x$, $y \in V_x$ and $U_x \cap V_x=\phi$. Hence $\{U_x: x \in X-U\}$ is a T_2 open cover of $X-U$ which is T_2 compact, and so there is a finite subcover

$$U_{x_1}, \cdots, U_{x_n}. \qquad \text{Let } V=\cap\{V_{x_i}: i=1, \cdots, n\}$$

It is now easy to prove that either of the metrics d_1 and d_2, given by

$$d_1(x, y)=\frac{1}{2}\{p(x, y)+q(x, y)\} \quad \text{and}$$

$$d_2(x, y)=\max\{p(x, y),\ q(x, y)\} \qquad \text{for } x, y \in X,$$

induces the topology T_1, so that (X, T_1) is metrizable.

The question now arises as to whether the compactness condition of Theorem 1 can be relaxed.

Example 1. This is a modification of an example due to Balanzat [1]. Let X be the set of positive integers and define the non negative real valued function q on $X \times X$ by