132. On the Convergence of the Godounov's Scheme for First Order Quasi Linear Equations

By Alain Yves LE ROUX
Départment de Mathématiques, I. N. S. A. 35000, Rennes, France

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Let $T>0$, $u_0 \in L^\infty(\boldsymbol{R})$, which is assumed of locally bounded variation; we consider the Cauchy's problem:

$$(1) \qquad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}[f(u,x,t)]+g(u,x,t)=0 \qquad \text{if } (x,t)\in \boldsymbol{R}\times]0,T[;$$

$$(2) \qquad u(x,0)=u_0(x) \qquad \text{if } x\in \boldsymbol{R};$$

where $f\in C^1(\boldsymbol{R}^2\times]0,T[)$, $g\in C^0(\boldsymbol{R}^2\times]0,T[)$ are such that g, f and $\partial f/\partial x$ are Lipschitz continuous with respect to u, uniformly in $(x,t)\in\boldsymbol{R}\times]0,T[$, g and $\partial f/\partial x$ are Lipschitz continuous with respect to x, uniformly in $(u,t)\in\boldsymbol{R}\times]0,T[$, and for $u=0$, $g(0,\cdot,\cdot)$ and $\partial f/\partial x(0,\cdot,\cdot)$ are uniformly bounded on $\boldsymbol{R}\times]0,T[$.

The problem (1), (2) is generally non linear: the solution may be discontinuous and not unique, so we need a weak definition.

Definition 1. A *weak solution* of (1), (2) is a function $u\in L^\infty(\boldsymbol{R}\times]0,T[)$, satisfying:

$$(3) \quad \iint_{\boldsymbol{R}\times]0,T[}\left\{u\frac{\partial\phi}{\partial t}+f(u,x,t)\frac{\partial\phi}{\partial x}-g(u,x,t)\phi\right\}dxdt+\int_{\boldsymbol{R}}\phi(x,0)u_0(x)dx=0,$$

for any $\phi\in C^2(\boldsymbol{R}\times]0,T[)$, with compact support.

The existence of a weak solution can be proved by the vanishing viscosity method, from the parabolic equation with $\varepsilon>0$:

$$(4) \qquad \frac{\partial u_\varepsilon}{\partial t}+\frac{\partial}{\partial x}[f(u_\varepsilon,x,t)]+g(u_\varepsilon,x,t)=\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2},$$

using a compactness argument in $L^1_{\mathrm{loc}}(\boldsymbol{R}\times]0,T[)$ for the family $\{u_\varepsilon\}_{\varepsilon>0}$ (see [3]).

But uniqueness of weak solutions of (1), (2), is not ensured; starting from (4) rather than (1), Kruzkov proposes another definition of solutions, that makes existence and uniqueness sure. See [3], and Hopf [2].

Definition 2. A Kruzkov's solution of (1), (2) is a function $u\in L^\infty(\boldsymbol{R}\times]0,T[)$, satisfying:

$\forall k\in\boldsymbol{R}$, $\forall\phi\in C^2(\boldsymbol{R}\times]0,T[)$, with compact support and non negative:

$$(5) \quad \begin{aligned}\iint_{\boldsymbol{R}\times[0,T[}\Big\{&|u-k|\frac{\partial\phi}{\partial t}+sg(u-k)(f(u,x,t)-f(k,x,t))\frac{\partial\phi}{\partial x}\\ &-sg(u-k)\left(\frac{\partial f}{\partial x}(k,x,t)+g(u,x,t)\right)\phi\Big\}dxdt\geq 0,\end{aligned}$$

where sg is the sign function: $sg(x)=x/|x|$ if $x\neq 0$, $sg(0)=0$. $\forall R>0\ \exists\mathcal{E}$