149. On a Pair of Groups and its Sylow Bases

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Only finite groups are to be considered in this note. Any unexplained notation and terminology should be referred to [1] and [2]. Throughout this note, let A and B be groups such that a pair $(A, B: f, g)$ of groups is well defined, where $f: A \to B$ and $g: B \to A$ are homomorphisms and let $|A|=|B|=p_1^{e_1}\cdots p_n^{e_n}$, where the p's are different primes and each e_i is a positive integer. Suppose A is solvable. Then B is also solvable. In this case, we shall call $(A, B: f, g)$ *solvable*. By P. Hall ([3]), the classical theorems about Sylow subgroups have been extended to the Sylow systems of a solvable group. With respect to $(A, B: f, g)$ which is solvable, we will give the following which are analogous to P. Hall's results. We denote by $\{S_i\}_n$ ($\{T_i\}_n$) a set of Sylow p_i-subgroups $S_i(T_i)$ of $A(B)$, $i=1, \cdots, n$, respectively.

Theorem 1. *Let $(A, B: f, g)$ be solvable and $\{S_i\}_n$ a Sylow basis of A. Then there is a Sylow basis $\{T_i\}_n$ of B such that for each $i=1, \cdots, n$, $(S_i, T_i: f, g)$ is well defined.*

The set $\{(S_i, T_i: f, g)\}_n$ given in Theorem 1 is called a *Sylow basis* of $(A, B: f, g)$.

Theorem 2. *Let $(A, B: f, g)$ be solvable, let $(M, N: f, g)$ be a subgroup of $(A, B: f, g)$ and $\{(P_i, Q_i: f, g)\}_m$ with $m \leqslant n$ a Sylow basis of $(M, N: f, g)$, where each P_i has order a power of p_i. Then there is a Sylow basis $\{(S_i, T_i: f, g)\}_n$ of $(A, B: f, g)$ such that for each $i=1, \cdots, m$, $(M \cap S_i, N \cap T_i: f, g)$ is well defined and equal to $(P_i, Q_i: f, g)$.*

Corollary. *Let $(A, B: f, g)$ be solvable and let $\{(S_i, T_i: f, g)\}_m$ with $m \leqslant n$ be a set of Sylow p_i-subgroups $(S_i, T_i: f, g)$ of $(A, B: f, g)$, $i=1, \cdots, m$, such that for each $i, j=1, \cdots, m$, $S_iS_j=S_jS_i$ and $T_iT_j=T_jT_i$. Then there is a Sylow basis $\{(S_i, T_i: f, g)\}_n$ of $(A, B: f, g)$ which contains $\{(S_i, T_i: f, g)\}_m$.*

To prove those theorems, we prepare some lemmas. Let π denote a set of primes and $(M, N: f, g)$ a subgroup of $(A, B: f, g)$ such that M is a π-subgroup (a Hall π-subgroup) of A. Then N is also a π-subgroup (a Hall π-subgroup) of B. In this case, we shall call $(M, N: f, g)$ a *π-subgroup* (a *Hall π-subgroup*) of $(A, B: f, g)$. The following is well known.

Lemma 1. *Let H be a Hall π-subgroup of a solvable group A and $M \lhd A$. Then $H \cap M$ and MH/M are Hall π-subgroups of M and A/M,*