10. A Remark on Shirota's Theorem

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Let $\varDelta$ be an arbitrary nonempty set and $R^{\varDelta}$ the Tihonov semifield, i.e. the ring of all real functions $\varDelta \to R$, taken in the product topology and with the natural partial order. Next let X be a set. A function $\rho: X^2 \to R^{\varDelta}$ is called a metric in X over $R^{\varDelta}$ (see [3]) if it satisfies the usual axioms for a pseudometric.

The metrics over topological semifields are convenient tools for considering uniformity, proximity, and topology through a viewpoint analogous to that of classical metric spaces. For example, this viewpoint has had some reflections in a series of works written by K. Iséki and S. Kasahara [4]–[8].

The purpose of this article is to formulate the Shirota's theorem in the language of metric space, which henceforth we shall call generalized metric space (GMS), and show some of the possible generalizations.

Let (X, ρ) be a given GMS, and let t_ρ denote the natural topology generated by ρ. We say that ρ is complete or Weil-complete if every ξ-sequence is convergent (see [3]). Then there naturally arises the following

Question A. Is there for any metric ρ *another* metric ρ' such that $t_\rho = t_{\rho'}$ and ρ' is complete?

In the case where $|\varDelta| = 1$, i.e. ρ is a usual real metric on X, we have the following theorem, which answers Question A in the positive.

Theorem (Gillman and Jerison [1]). *For any real metric ρ; $X^2 \to R$, there exists a complete metric $\rho': X^2 \to R^{\varDelta'}$ such that $t_\rho = t_{\rho'}$.*

To give for an answer to Question A in the case where $\varDelta$ is arbitrary, we consider the following well-known process for constructing the metric $\rho' = H(\rho)$ from the given metric ρ (see Gillman and Jerison [1]).

If $\rho: X^2 \to R^{\varDelta}$ is given, then $\rho' = H(\rho): X^2 \to R^{\varDelta'}$ is defined by

$$\rho'(x, y) \cdot q' = |q'(x) - q'(y)|,$$

where $\varDelta' = C(X, t_\rho)$ is the family of all continuous real functions on the topological space (X, t_ρ).

It is easy to see that $t_\rho = t_{\rho'}$. Now we reduce Question A to the following question about the metric $H(\rho)$.

Question A′. Under what conditions on the GMS (X, ρ), is the