6. A Note on the Large Sieve

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1. Let $a(n)$ be arbitrary complex numbers. Let $c_r(n)$ and $\varphi(n)$ be the Ramanujan sum and the Euler function, respectively. Then a slight modification of a recent large sieve inequality of Selberg [1; Théorème 7A] states that we have, uniformly for any Q, R, M, N, k, l,

$$(\sharp)\qquad \sum_{\substack{q\leqq Q,\ r\leqq R\\ (q,r)=(qr,k)=1}} \frac{q}{\varphi(qr)} \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{M\leqq n<M+N\\ n\equiv l\ (\mathrm{mod}\ k)}} \chi(n)c_r(n)a(n)\Bigg|^2 \leqq (N/k+(QR)^2) \sum_{\substack{M\leqq n<M+N\\ n\equiv l\ (\mathrm{mod}\ k)}} |a(n)|^2,$$

where $\sum^*$ denotes as usual a sum over primitive characters χ (mod q). This, in case of $k=1$, has an important application to Dirichlet's L-functions (see [1; p. 40 and p. 83] and also [6] [3]), but in the present note we are concerned with its sieve-effect. As is easily seen, ($\sharp$) implies the linear sieve result of Bombieri-Davenport [1; Théorème 8] and thus the Brun-Titchmarsh theorem. On the other hand the B-T theorem has recently got some improvements (see [4] and also [5] [2] [7]). So, noticing the fact that the dual of ($\sharp$), in case of $Q=1$, is by virtue of $c_r(n)$ reduced to the form similar to the classical sieve idea of Selberg, we may well expect that ($\sharp$) can be improved so as to contain our improvements of the T-B theorem. Then we shall have a first example of large sieve inequalities sensitive to arithmetic progressions.

Now we announce such an improvement of ($\sharp$):

Theorem. *If $(k, l)=1$, then we have*

$$\sum_{\substack{q\leqq Q,\ r\leqq R\\ (q,r)=(qr,k)=1}} \frac{q}{\varphi(qr)} \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{n\leqq N\\ n\equiv l\ (\mathrm{mod}\ k)}} \chi(n)c_r(n)a(n)\Bigg|^2 \leqq \Lambda \sum_{\substack{n\leqq N\\ n\equiv l\ (\mathrm{mod}\ k)}} |a(n)|^2,$$

where, ε being an arbitrary small positive constant,

$$\Lambda=\frac{N}{k}(1+O((\log N)^{-1}))+O_\varepsilon\Big(\frac{QR^{1+\varepsilon}}{\sqrt{k}}(R+kQ^2)(\log N)^4\Big).$$

Corollary. *Let p denote a prime and let $\pi(N; k, l)$ be the number of primes $\equiv l$ (mod k) less than N. Then we have, under the condition $N^{2/5}\geqq Q^2k$,*

$$\sum_{\substack{q\leqq Q\\ (q,k)=1}} \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)} \Bigg| \sum_{\substack{p\equiv l\ (\mathrm{mod}\ k)\\ p\leqq N}} \chi(p)\Bigg|^2 \leqq (2+\varepsilon)\frac{N}{\varphi(k)\log(N/(\sqrt{k}\,Q))}\pi(N; k, l).$$