2. *A Remark on Fractional Powers of Linear Operators in Banach Spaces*

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1. Introduction. Let X be a Banach space and A be a densely defined, closed linear operator in X satisfying

(1) the resolvent set $\rho(-A)$ of $-A$ contains the non-negative real axis and

(2) $$\|\lambda(\lambda+A)^{-1}\| \leqq M \qquad \text{for } \lambda>0,$$

or equivalently,

$(2)_1$ $$\|\lambda(\lambda+A)^{-1}\| \leqq M_1 \qquad \text{for } |arg\lambda| \leqq \omega$$

holds, where M, M_1 and ω are some positive constants independent of λ. As is well known, the fractional power A^α, $0<\alpha<1$ of A is defined through

(3) $$A^{-\alpha}=\frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda-A)^{-1}d\lambda,$$

where Γ runs in $\rho(A)$ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ $(\pi-\omega\leqq\theta\leqq\pi)$ avoiding the non-positive real axis.

The purpose of the present paper is to describe a criterion for the width of the domain $D(A^\alpha)$ of A^α, and then apply it to an evolution equation of parabolic type:

$$du(t)/dt+A(t)u(t)=f(t), \qquad 0\leqq t\leqq T.$$

2. Basic theorem. We denote by $D(A_\alpha)$, $0<\alpha<1$ the set of all $x\in X$ such that $\int_\Gamma \lambda^{\alpha-1}A(\lambda-A)^{-1}xd\lambda$ is absolutely convergent and define a linear operator A_α by

(4) $$A_\alpha x=\frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1}A(\lambda-A)^{-1}xd\lambda, \qquad x\in D(A_\alpha).$$

In view of (3) it is evident that $D(A)$ is contained in $D(A_\alpha)$, $0<\alpha<1$.

Lemma. *If $x\in X$ and*

$(5)_1$ *$\lambda^\beta A(\lambda+A)^{-1}x$, $|arg\lambda|\leqq\omega$ is uniformly bounded for some $0<\beta\leqq 1$, then $x\in D(A^\alpha)$ and $A^\alpha x=A_\alpha x$ for any α with $0<\alpha<\beta$.*

Proof. Clearly $x\in D(A_\alpha)$, $0<\alpha<\beta$ and (4) holds good. From

$$A^{-1}A^{-\alpha}A_\alpha x=A^{-\alpha}A^{-1}A_\alpha x=A^{-\alpha}\frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1}(\lambda-A)^{-1}xd\lambda$$
$$=A^{-\alpha}A^{\alpha-1}x=A^{-1}x,$$

it follows that $A^{-\alpha}A_\alpha x=x$, which implies that $x\in D(A^\alpha)$ and $A^\alpha x=A_\alpha x$.

Theorem. *Let A be a densely defined, closed linear operator satis-*