36. On the Existence of Invariant Functions for Markov Representations of Amenable Semigroups

By Kôkichi SAKAI
College of Liberal Arts, Kagoshima University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1977)

§ 0. **Introduction.** Let (X, Σ, m) be a σ-finite measure space and S be a left amenable semigroup. By L^1 and L^∞ we denote the usual Banach spaces $L^1(X, \Sigma, m)$ and $L^\infty(X, \Sigma, m)$ respectively. Let $\boldsymbol{T} = \{T_s; s \in S\}$ be a representation of S by positive linear contractions on L^1. For the sake of brevity such $\boldsymbol{T}$ is called a *Markov representation* of S on L^1. By $co(\boldsymbol{T})$ we denote the convex hull of $\{T_s; s \in S\}$ and by $\overline{co}(\boldsymbol{T})$ the closure of $co(\boldsymbol{T})$ with respect to the operator norm topology. For this $\boldsymbol{T}$ we consider the following conditions:

(A) *There exists a strictly positive function f in L^1 such that $T_s f = f$ for all $s \in S$.*

(B) *Every operator in $\overline{co}(\boldsymbol{T})$ is conservative.*

(C) *T_s is conservative for every $s \in S$.*

Then it is obvious that the condition (A) implies (B) and (C). In this paper we shall prove the next theorems.

Theorem 1. *For any Markov representation $\boldsymbol{T} = \{T_s; s \in S\}$ of a left amenable semigroup S on L^1, the conditions* (A) *and* (B) *are mutually equivalent.*

Theorem 2. *Let S be an extremely left amenable semigroup. Then for any Markov representation $\boldsymbol{T} = \{T_s; s \in S\}$ of S on L^1 with the following property:*

(1) $\quad T_s^*(gh) = T_s^*(g) T_s^*(h) \quad$ *for any $g, h \in L^\infty$ and $s \in S$,*

the conditions (A) *and* (C) *are mutually equivalent.*

Theorem 1 is proved by Brunel [1] for the case when S is the additive semigroup of positive integers, and by Horowitz [3] for the case when S is commutative. In the author's paper [4] we shall show that the main theorem in [3] is also valid for the case of left amenable semigroups of Markov operators.

§ 1. **Proof of Theorem 1.** Let S and $\boldsymbol{T} = \{T_s; s \in S\}$ be as in Theorem 1. By $L(\boldsymbol{T})$ we denote the closed linear subspace of L^∞ generated by $\{T_s^* h - h; s \in S, h \in L^\infty\}$ and put $L^+(\boldsymbol{T}) = \{h \in L(\boldsymbol{T}); h \geq 0\}$. Then the next lemma is well-known (e.g., see Granirer [2, Theorem 5]).

Lemma 3. *For any $f \in L^\infty$ the following equality holds:*

(2) $\quad \inf \{\|f - h\|_\infty; h \in L(\boldsymbol{T})\} = \inf \{\|Q^* f\|_\infty; Q \in co(\boldsymbol{T})\}.$

Especially if S is extremely left amenable, then