33. A Note on Malmquist's Theorem on First Order Algebraic Differential Equations

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The entitled theorem reads as follows: *If the differential equation*

(1) $dw/dz=R(z,w)$ (*R is a rational function of z and w*)

has a transcendental meromorphic solution $w(z)$, then the equation must be of the Riccati type, i.e., $R(z,w)$ must be a polynomial of the second degree in w.

In 1933 the present author gave, as an application of the Nevanlinna theory ([5]) of meromorphic functions, another proof of this striking theorem of J. Malmquist [4] dating 1913. In this proof (Yosida [9] and [10]), a decisive role was played by a theorem of G. Valiron [6]:

(2) $$T(r,R(z,w(z)))^{1)}=d\cdot T(r,w(z))+O(\log r),$$

where d is the degree in w of $R(z,w)$. In 1950, H. Wittich ([7] and [8]) gave an alternate proof which is based upon the fact that the order of the meromorphic function $w(z)$ is finite and that its proximity function $m(r,w(z))$ is $O(\log r)$. Recently in 1974, E. Hille ([2] and [3]) gave another approach proposing a geometric argument instead of Wittich's estimation via the calculus of residues. It is to be noted here that, for the finiteness of the order of the meromorphic solution $w(z)$, the author gave in 1934 a straightforward proof ([10], Theorem 7) relying upon the T. Shimizu-L. Ahlfors-H. Cartan interpretation (see, e.g., [5], 165–) of the Nevanlinna characteristic $T(r,w(z))$.

In view of the above, I should like to show that my original idea in [9] and [10] can be pursued to the result without appealing to the theorem of Valiron nor to the Wittich-Hille type estimation.

We may assume that

(3) $$R(z,w)=P(z,w)/Q(z,w)=\left(\sum_{j=0}^{p}p_j(z)w^j\right)/\left(\sum_{k=0}^{q}q_k(z)w^k\right)$$

with polynomial coefficients p_j's and q_k's such that $p_p(z)\cdot q_q(z)\not\equiv 0$ and w-polynomials $P(z,w)$ and $Q(z,w)$ have no factos in common. By virtue of the *defect relation* in the Nevanlinna theory, we have

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1) We shall follow notations in [1]:

$T(r,f(z))=m(r,f(z))+N(r,f(z))$, $m(r,f(z))=(2\pi)^{-1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$, $N(r,f(z))=\int_0^t t^{-1}(n(t,f(z))-n(0,f(z)))dt+n(0,f(z))\cdot\log r$, where $n(r,(f(z))$ denotes the number of poles of $f(z)$ for $|z|\leqq r$, multiple poles being counted with the multiplicity.