# 32. On Multivalent Functions in Multiply Connected Domains. I

By Hitoshi ABE
Department of Applied Mathematics, Faculty of Engineering, Ehime University

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**1. Introduction.** E. Rengel [3] derived many results on univalent functions in the multiply connected, representative domains (defined hereafter) by means of the so-called Rengel's inequality. In this paper we shall deal with multivalent functions in multiply connected domains by means of the fundamental inequalities based on the extremal length method which are extensions of Rengel's inequality (cf. [2] or [4]).

We shall first define the $n$-ply connected, representative domains as follows.

$D_1$: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ circular arc slits centered at the origin.

$D_2$: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ radial slits emanating from the origin.

$D_3$: the unit circle with $(n-1)$ circular arc slits centered at the origin.

$D_4$: the unit circle with $(n-1)$ radial slits emanating from the origin.

$D_5$: the whole plane with $n$ circular arc slits centered at the origin.

$D_6$: the whole plane with $n$ radial slits emanating from the origin.

We shall define circumferentially mean $p$-valent functions in a domain $D$, according to Biernacki (cf. Hayman [1]).

Let $n(R,\Phi)$ denote the number of roots of the equation $f(z)=w=\mathrm{Re}^{i\Phi}$. We define $p(R)$ as follows.

$$p(R)=\frac{1}{2\pi}\int_0^{2\pi} n(R,\Phi)d\Phi \qquad (0\le R<\infty). \tag{1.1}$$

If $p(R)\le p$ $(0\le R<\infty)$, $f(z)$ is called "circumferentially mean $p$-valent". In this paper we assume that $p$ is a positive integer.

**2. Fundamental inequalities.** **Theorem 2.1.** *Let $f(z)$ be single-valued, regular, circumferentially mean p-valent in $D_1$ and satisfy the condition $\int_C |d\arg f(z)|\ge 2\pi p(C: |z|=r\ (r_1<r<r_2))$ where the circle $C$ does not contain any circular slit of $D_1$. Then we have the following inequality*