30. *Some Derived Rules of Intuitionistic Second Order Arithmetic*

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L. E. J. Brouwer introduced some intuitionistic principles in his study of intuitionistic analysis. They cannot be proved in the intuitionistic second order arithmetic. Moreover some of them are incompatible with the classical mathematics when they are interpreted in the classical sense. But it has been shown, that the derived rules which correspond to some of Brouwer's principles are valid for various intuitionistic systems (e.g. [1], [2]).[1] The purpose of this note is to announce the result that the derived rules which correspond to Brouwer's principles are valid for the intuitionistic second order arithmetic. Details will be published elsewhere. The author would like to thank Professor Troelstra for his helpful suggestion.

Let $\boldsymbol{S}$ be the formal system of intuitionistic second order arithmetic with equality (i.e. ***HAS*** of [1]). In this system one can define the notions of functions and real numbers in the sense of Cauchy sequence of rational numbers. We use $f, g, h, \cdots$ as the variables for functions from natural numbers to natural numbers, and use $x, y, z, m, n, \cdots$ as the variables for natural numbers. $A, B, \cdots$ stand for formulae of $\boldsymbol{S}$. $\bar{f}(x)$, $(h|f)$, $f(g)$, $!\,f(g)$, $!\,(h|f)$, and $f \leq g$ will be used in the following sense:

$$\begin{cases} \bar{f}(0) =_{\mathrm{def}} \langle\ \rangle \\ \bar{f}(n+1) =_{\mathrm{def}} \langle f(0), \cdots, f(n)\rangle \end{cases}$$

$$(h|f)(x) \simeq y \equiv_{\mathrm{def}} h(\langle x\rangle * \bar{f}(\min_z [h(\langle x\rangle * \bar{f}(z)) > 0])) \dot{-} 1 = y$$

$$f(g) \simeq y \equiv_{\mathrm{def}} f(\bar{g}(\min_z [f(\bar{g}(z)) > 0])) - 1 = y$$

$$!\,f(g) \equiv_{\mathrm{def}} \exists y (f(g) \simeq x)$$

$$!\,(h|f) \equiv_{\mathrm{def}} \forall x \exists y (y \simeq (h|f)(x))$$

$$f \leq g \equiv_{\mathrm{def}} \forall x (f(x) \leq g(x)).$$

Theorem 1 (Continuity Rule). *If* $\boldsymbol{S} \vdash \forall f \exists g A(f, g)$, *then* $\boldsymbol{S} \vdash \exists h$ $\{h$ *is primitive recursive* & $\forall f(!\,(h|f)$ & $A(f, (h|f))\}$.

Theorem 2 (Fan Rule). *If* $\boldsymbol{S} \vdash \forall f \exists n A(f, n)$, *then* $\boldsymbol{S} \vdash \exists f[f$ *is primitive recursive* & $\forall g\{!\,f(g) \& (\forall h \leq g \exists n \forall k(\bar{h}(f(g)) = \bar{k}(f(g)) \to A(k, n))\}]$.

Theorem 3. *Let* $\boldsymbol{R}$ *be the whole of real numbers.* (i) *If* $\boldsymbol{S} \vdash \{A$ *is*

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1) Professor Troelstra informed the author that the Bar Induction Rule for $\boldsymbol{HA}^{\omega}$ has been obtained by H. Schwichtenberg in his unpublished inaugural dissertation (1973).