38. *On Hankel Transforms.*

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(Comm. by M. FUJIWARA, M.I.A., March 12, 1929.)

1. The notion of the Fourier transforms arises from Fourier's integral formula

$$(1) \qquad f(x)=\frac{2}{\pi}\int_0^\infty \cos x\,u\,du\int_0^\infty \cos x\,t f(t)\,dt,$$

which gives the reciprocal relation

$$(2) \quad f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos x\,u\,F(u)\,du\,, \quad F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos x\,u f(u)\,du$$

between two functions $f(x)$ and $F(x)$. Each one of two functions so related is said to be the Fourier transform of the other. When we study the relations (2), it is desirable to find any theorem of the form: "when $f(x)$ satisfies certain conditions, so does $F(x)$, and the reciprocity holds goods." For this purpose the reciprocal relations above stated were tranformed into the from

$$(3) \quad f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}F(u)du\,, \quad F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}f(u)du\,,$$

which reduce to (2), when differentiation under the sign of integration is permissible. Then the following theorem was established.

If $\int_0^\infty (f(x))^2\,dx$ *exists, then* $\int_0^\infty (F(x))^2\,dx$ *exists, and the reciprocity holds. Further we have* $\int_0^\infty (f(x))^2\,dx=\int_0^\infty (F(x))^2\,dx$.

This was gotten by Prof. Plancherel.[1] He deduced this theorem as a special case of the reciprocal relation belonging to a general type of functional transformation. But the deduction is by no means immediate. Lately Mr. Titchmarsh[2] proved this theorem in direct way, and Prof. Hardy[3] gave an alternate proof.

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1) Plancherel: Rendiconti di Palermo, **30** (1910).
2) Titchmarsh: Proc. Cambridge Phil. Soc., **21** (1924).
3) Hardy: Messenger of Mathematics, **48** (1925).