PAPERS COMMUNICATED

40. *On a Condition of Stability for a Differential Equation.*

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(Rec. Mar. 24, 1930. Comm. by T. YOSIE, M.I.A., April 12, 1930.)

Recently O. Perron[1] has pointed out the inaccuracy of Fatou's criterion for stability in relation to the differential equation

$$\frac{d^2x}{dt^2}+\Phi(t)x=0\,, \tag{1}$$

where $\Phi(t)$ denotes a continuous real function lying between the positive boundaries $a^2\leqq\Phi(t)\leqq b^2$ for all values of t.

Fatou asserted that the integrals of the differential equation (1) and their derivatives are bounded, while Perron gave an example having an integral not bounded even when $\lim_{t\to\infty}\Phi(t)=1$.

Fatou's assertion may however be amended in the following manner:

If the improper integral $\int_{t_0}^{\infty}|\Phi(t)-c^2|\,dt$ *converges, where* c *is a positive constant, then the integrals of the differential equation* (1) *and their first derivatives are bounded for* $t>t_0$.

Proof: Consider the integral $x(t)$ of (1) and the integral $y(t)$ of the differential equation

$$\frac{d^2y}{dt^2}+c^2y=0\,, \tag{2}$$

with the same initial values for $t=t_1\,(>t_0)$. From (1) and (2) we obtain the identity

$$\frac{d^2}{dt^2}(x-y)+c^2(x-y)=(c^2-\Phi)\,x\,,$$

and hence

$$x-y=\frac{1}{c}\left\{\sin ct\int_{t_1}^{t}(c^2-\Phi)x\cos ct\,dt-\cos ct\int_{t_1}^{t}(c^2-\Phi)x\sin ct\,dt\right\}. \tag{3}$$

As it is always possible from our assumption, let us now take t_1 so large that

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1) O. Perron, Über ein vermeintliches Stabilitätskriterium, Gött. Nachr. (1930), 1.