94. On the Distribution of a-points of Solutions for Linear Differential Equation of the Second Order.

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Consider two differential equations

$$y'' = \{F(x) + iG(x)\}y, \tag{1}$$

$$Z'' = H(x)Z, \tag{2}$$

where $F(x)$, $G(x)$ and $H(x)$ are real and continuous functions in the domain $D: 0 \leqq x \leqq c$, such that $H(x) \leqq F(x)$ in D.

We put $|y(x)| = R(x)$ and $\arg y(x) = \theta(x)$, then $R(x)$ will satisfy the differential equation

$$R'' = \{F(x) + \theta'(x)^2\}R. \tag{3}$$

Lemma. Let the initial values of $R(x)$, $Z(x)$ be given by

$$\begin{cases} R(0) = Z(0) = |a|, \text{ a being a constant, real or imaginary,} \\ R'(0) = Z'(0) > 0. \end{cases} \tag{4}$$

If we denote by x_1 and x_2 the next $|a|$—points to $x=0$ in D of $R(x)$ and $Z(x)$ respectively, then we must have

$$x_1 \geqq x_2. \tag{5}$$

Proof. By (4), $R(x)$ and $Z(x)$ are both $>|a|$ for $0 < x < \min(x_1, x_2)$. From (2) and (3) we obtain $\frac{d}{dx}(R'Z - RZ') = (\theta'^2 + F - H)RZ$. Integrating from 0 to x and remembering (4), we have

$$\begin{cases} R'(x)Z(x) - R(x)Z'(x) = \int_0^x (\theta'^2 + F - H)RZdx \\ \qquad \geqq \int_0^x (\theta'^2 + F - H)a^2dx \geqq 0, \\ 0 \leqq x \leqq \min(x_1, x_2). \end{cases} \tag{6}$$

Therefore we shall have $R(x) \geqq Z(x) \geqq |a|$ for $0 \leqq x \leqq \min(x_1, x_2)$. Hence, if it be possible that $x_1 < x_2$, we should have $|a| = R(x_1) \geqq Z(x_1) \geqq |a|$, that is $Z(x_1) = |a|$ or $x_1 = x_2$. This is a contradiction, and so we must have $x_1 \geqq x_2$.

Remark. The equality sign of (5) holds if and only if $\theta'(x) \equiv 0$ and $F(x) \equiv H(x)$ in $[0, x_2]$.