# PAPERS COMMUNICATED

## *108. On the Wiener's Formula.*

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*1.* Wiener[1] has proved that:

If $$\mathfrak{M}\{f\}=\lim_{x\to\infty}\frac{1}{x}\int_0^x f(\xi)d\xi$$

exists and is finite and $\frac{1}{x}\int_0^x |f(\xi)|d\xi$ is bounded in $(0, \infty)$, then

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_0^\infty f(x)\frac{\sin^2\varepsilon x}{\varepsilon x^2}dx=\mathfrak{M}\{f\}. \tag{1}$$

Bochner[2] has replaced the kernel $\frac{\sin^2 x}{x^2}$ in (1) by a general function $K(x)$ and found the conditions for the validity of

$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx=\mathfrak{M}\{f\}\int_0^\infty K(x)dx. \tag{2}$$

Bochner named (2) the Wiener's formula.

In this paper, we treat the conditions of validity of (2).

*2. Theorem 1.* Suppose that (i) $K(x)$ is absolutely continuous in any finite interval, (ii) $K(x)$ is absolutely integrable in $(0, \infty)$, (iii) $xK(x)$ is of bounded variation in $(0, \infty)$, and (iv) $\frac{1}{x}\int_0^x f(\xi)d\xi$ is bounded in $(0, \infty)$, and (v) the limit $\mathfrak{M}\{f\}=\lim_{x\to\infty}\frac{1}{x}\int_0^x f(\xi)d\xi$ exists and is finite. Then we have

$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx=\mathfrak{M}\{f\}\int_0^\infty K(x)dx. \tag{2}$$

*Proof.* Without loss of generality, we can suppose that

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1) Wiener, Math. Zeits., **24** (1926); ——, Journ. Math. and Phys. M. I. T., **5** (1926); ——, Journ. London Math. Soc., **2** (1927). *Cf.* Bochner-Hardy, Journ. London Math. Soc., **1** (1926); Jacob, Journ. London Math. Soc., **3** (1928); Littauer, Journ. London Math. Soc., **4** (1929); Wiener, Acta Math., **55** (1931).

2) Bochner, Vorlesungen über Fouriersche Integrale, 1933, pp. 30–32.