153. Note on a Certain Multivalent Function.

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In this note we prove a theorem on a certain multivalent function.

Theorem. *Let*

$$w=f(z)=\frac{1}{z^k}+a_kz^k+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\cdots\cdots+a_nz^n+\cdots\cdots \qquad (a_k\neq 0)$$

be regular and k-valent in $0<|z|<1$, *then*

$$k|a_k|^2+(k+1)|a_{k+1}|^2+(k+2)|a_{k+2}|^2+\cdots\cdots+n|a_n|^2+\cdots\cdots\leqq k\,.$$

Proof. We consider at first a circle $|z|=r$ $(0<r<1)$, then we may write

$$\Big|\sum_{n=k+1}^{\infty}a_nz^{n-k}\Big|<\delta\,,$$

where δ denotes a certain positive constant. Therefore, if we write

$$\zeta=\frac{1}{z^k}+a_kz^k$$

$$|w-\zeta|<\delta|z|^k=\delta r^k$$

and so

$$|w-2\sqrt{a_k}|+|w+2\sqrt{a_k}|\leqq 2|w-\zeta|+|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|$$
$$<2\delta r^k+|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|\,.$$

Now, since

$$|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|=|z^{\frac{k}{2}}-a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2+|z^{-\frac{k}{2}}+a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2$$
$$=2\{|z^{-\frac{k}{2}}|^2+|a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2\}$$
$$=2\{\frac{1}{r^k}+|a_k|r^k\}\,,$$

it follows that

$$|w-2\sqrt{a_k}|+|w+2\sqrt{a_k}|<2\Big\{\frac{1}{r^k}+(|a_k|+\delta)r^k\Big\}\,. \qquad (1)$$

Thus the image of $|z|=r$ by $w=f(z)$ lies in the elliptic domain (1) on the w-plane. Let A denote its area, then